Complex Motivic kq-resolutions

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2 Preliminaries

3 The E_1-term

4 Calculations

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6 A motivic J spectrum?

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*Everything in this talk is joint work with J.D. Quigley

Question

How can we compute the homotopy groups of the sphere spectrum S^0?

Motivation

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How can we compute the homotopy groups of the sphere spectrum S^0?

Adams found a very powerful technique. Let E be a commutative ring spectrum. Then we have a cofiber sequence

$$\Sigma^{-1}\overline{E} \to S^0 \to E \to \overline{E}.$$

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Adams found a very powerful technique. Let E be a commutative ring spectrum. Then we have a cofiber sequence

$$\Sigma^{-1}\overline{E} \to S^0 \to E \to \overline{E}.$$

Adams' idea was to keep smashing this cofiber sequence with $\overline{E}$. This yields a tower of cofibrations.

Motivation

This is what the tower looks like.

$$\begin{tikzcd}
S^0 & \Sigma^{-1}\overline{E} \arrow[l] \arrow[d] & \Sigma^{-2}\overline{E}^{\wedge 2} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
E \arrow[from=u] & \Sigma^{-1}E \wedge \overline{E} & \Sigma^{-2}E \wedge \overline{E}^{\wedge 2} &
\end{tikzcd}$$

Motivation

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$$\begin{tikzcd} S^0 \arrow[d] & \Sigma^{-1}\overline{E} \arrow[l] \arrow[d] & \Sigma^{-2}\overline{E}^{\wedge 2} \arrow[l] \arrow[d] & \cdots \arrow[l] \\ E & \Sigma^{-1}E \wedge \overline{E} & \Sigma^{-2}E \wedge \overline{E}^{\wedge 2} & \end{tikzcd}$$

This gives rise to a spectral sequence, the E*-based ASS*, of the following form

$$E_1^{s,t} = \pi_t(\Sigma^{-s}E \wedge \overline{E}^{\wedge s}) \Longrightarrow \pi_{t-s}((S^0)^{\wedge}_E).$$

Motivation

This is what the tower looks like.

$$\begin{array}{ccccccc}
S^0 & \longleftarrow & \Sigma^{-1}\overline{E} & \longleftarrow & \Sigma^{-2}\overline{E}^{\wedge 2} & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
E & & \Sigma^{-1}E \wedge \overline{E} & & \Sigma^{-2}E \wedge \overline{E}^{\wedge 2} & &
\end{array}$$

This gives rise to a spectral sequence, the E*-based ASS*, of the following form

$$E_1^{s,t} = \pi_t(\Sigma^{-s} E \wedge \overline{E}^{\wedge s}) \implies \pi_{t-s}((S^0)^{\wedge}_E).$$

Under favorable circumstances, we even know the E_2-term as

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*).$$

Motivation

Examples

1. When $E = H\mathbb{F}_p$ this recovers the classical Adams spectral sequence.
2. When $E = MU$ or $E = BP$, then this the Adams-Novikov spectral sequence.

In both instances the E_2-term is describable as an Ext group.

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Mahowald intensively studied the case when $E = ko$, the connective cover of real K-theory KO. In this case there is no nice description of the E_2-term.
However, Mahowald was able to do a lot with the ko-based ASS. Most notably he used it to find the v_1-periodic elements in π_*S^0 and prove the height 1 telescope conjecture.

Motivation

The construction of Adams spectral sequences is very general; it can be done in any tensor triangulated category $\mathscr{C}$. In particular when $\mathscr{C} = \mathsf{SH}(S)$ is the stable motivic homotopy category over a base scheme S.

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In $\mathrm{SH}(S)$, there is an analogue of KO, referred to as Hermitian K-theory and denoted as KQ. There is also a "connective version" called kq.

Question

What does the kq-based Adams spectral sequence look like over a base field F? What does the v_1-inverted version of this spectral sequence look like? What is this localized spectral sequence calculating?

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The construction of Adams spectral sequences is very general; it can be done in any tensor triangulated category $\mathscr{C}$. In particular when $\mathscr{C} = \mathrm{SH}(S)$ is the stable motivic homotopy category over a base scheme S.

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Question

What does the kq-based Adams spectral sequence look like over a base field F? What does the v_1-inverted version of this spectral sequence look like? What is this localized spectral sequence calculating?

In this talk I will discuss joint work with Quigley in the case $F = \mathbb{C}$.

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Basics on KQ and kq

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Everything is 2-complete. We let H stand for mod 2 motivic cohomology and we let $\mathbb{M}_2$ denote the cohomology of a point. So

$$\mathbb{M}_2 = \mathbb{F}_2[\tau] \qquad |\tau| = (0, -1)$$

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$$\mathbb{M}_2 = \mathbb{F}_2[\tau] \qquad |\tau| = (0, -1)$$

In order to compute the kq-based ASS the first thing one needs to know is the co-operations algebra $\pi_{**} kq \wedge kq$. A natural choice for this is the motivic Adams spectral sequence

$$\mathrm{Ext}_{A_{**}}(\mathbb{M}_2, H_{**} kq \wedge kq) \Longrightarrow \pi_{**} kq \wedge kq$$

So what is $H_{**}(kq)$?

$\mathbb{C}$-motivic Steenrod algebra

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The $\mathbb{C}$-motivic Steenrod algebra was computed by Voevodsky. It is given by

$$\mathscr{A}^{\mathbb{C}}_{**} \cong \mathbb{M}_2[\zeta_1, \zeta_2, \ldots; \overline{\tau}_0, \overline{\tau}_2, \ldots]/(\overline{\tau}_i^2 - \tau\zeta_{i+1}).$$

The bidegrees of the generators are given by

$$|\zeta_i| = (2^{i+1} - 2, 2^i - 1), \qquad |\overline{\tau}_i| = (2^{i+1} - 1, 2^i - 1).$$

$\mathbb{C}$-motivic Steenrod algebra

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$$|\zeta_i| = (2^{i+1} - 2, 2^i - 1), \qquad |\overline{\tau}_i| = (2^{i+1} - 1, 2^i - 1).$$

As in the classical case, this is a Hopf algebra. Its coproduct is given by the following

$$\psi(\zeta_k) = \sum_{i+j=k} \zeta_i \otimes \zeta_j^{2^i}, \quad \psi(\overline{\tau}_k) = \sum_{i+j=k} \overline{\tau}_i \otimes \zeta_j^{2^i} + 1 \otimes \overline{\tau}_k.$$

The homology of kq

We define kq to be the *very effective cover* of KQ. This was originally constructed by Isaksen-Shkembi. The homology of kq is known to be given by the following

$$H_{**}kq \cong A /\!\!/ A(1)_{**} \cong \mathbb{M}_2[\zeta_1^2, \zeta_2, \zeta_3, \ldots; \overline{\tau}_2, \overline{\tau}_3, \ldots]/(\overline{\tau}_i^2 - \tau\zeta_{i+1})$$

The homology is also a comodule over $A_{**}^{\mathbb{C}}$ via restricting the coaction.

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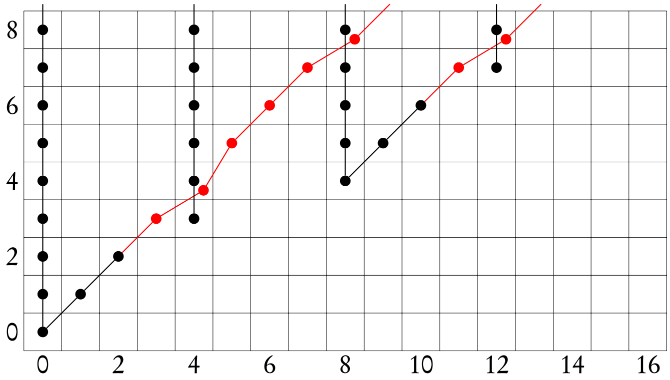
The homology is also a comodule over $A_{**}^{\mathbb{C}}$ via restricting the coaction.
Plugging this into the motivic Adams spectral sequence yields

$$\mathrm{Ext}_{A(1)_{**}}(\mathbb{M}_2, \mathbb{M}_2) \Longrightarrow \pi_{**}kq^{\wedge}_{(2,\eta)}.$$

$\pi_{**}kq$

Here is a picture of the Adams E_2-term for $\pi_{**}kq$.



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kq co-operations

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To study the kq-based ASS it is paramount to have some understanding of $\pi_{**}(kq \wedge kq)$. A natural candidate for this is the motivic Adams spectral sequence:

$$\mathrm{Ext}_{A^{\mathbb{C}}_{**}}(\mathbb{M}_2, H_{**}(kq \wedge kq)) \Longrightarrow \pi_{**}(kq \wedge kq).$$

By the change-of-rings isomorphism, the E_2-term is expressible as

$$\mathrm{Ext}_{A(1)^{\mathbb{C}}_{**}}(\mathbb{M}_2, A /\!\!/ A(1)_{**}).$$

Mahowald's approach to $ko \wedge ko$

Mahowald was able to understand $ko \wedge ko$ up to certain torsion classes.

Theorem (Mahowald)

There is an equivalence of spectra

$$ko \wedge ko \simeq \bigvee_{i \geq 0} \Sigma^{4i} ko \wedge H\mathbb{Z}_i^{cl}$$

where $H\mathbb{Z}_i^{cl}$ is the ith integral Brown-Gitler spectrum. Moreover, there is an equivalence

$$ko \wedge H\mathbb{Z}_i^{cl} \simeq \begin{cases} ko^{\langle 2i-\alpha(i)\rangle} \vee HV_i & i \equiv 0 \mod 2 \\ ksp^{\langle 2i-\alpha(i)-1\rangle} \vee HV_i & i \equiv 1 \mod 2 \end{cases}$$

Mahowald's approach to $ko \wedge ko$

Mahowald's approach is to first proceed at the level of homology: there is a homological shadow of the splitting.

Definition

Define the *Mahowald weight* on A_* by setting

$$\mathrm{wt}(\xi_i) = 2^{i-1}$$

and extending multiplicatively to monomials. We similarly define this on the subalgebras $A /\!/ A(n)_{**}$.

Proposition

The Mahowald weight defines a filtration by comodules on A_* and the subalgebras $A /\!/ A(n)_*$.

Mahowald's approach to $ko \wedge ko$

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We can identify the homology of the $H\mathbb{Z}_i^{cl}$ inside $A/\!/A(1)_*$.

Proposition

There are $A(1)_*$-isomorphisms

$$H_*H\mathbb{Z}_i^{cl} \cong (A/\!/A(1)_*)^{\mathrm{wt}=4i}$$

and there is an $A(1)_*$-isomorphism

$$A/\!/A(1)_* \cong \bigoplus_{i \geq 0} (A/\!/A(1)_*)^{\mathrm{wt}=4i} \bigoplus_{i \geq 0} H_*H\mathbb{Z}_i^{cl}.$$

kq co-operations continued

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Our idea was to mimic Mahowald's work on $ko \wedge ko$. The problem is that there is currently no construction of motivic analogues of Brown-Gitler spectra. Instead, we proceed algebraically.

Definition

Define the *Mahowald weight* on $A^{\mathbb{C}}_{**}$ by setting

$$\mathrm{wt}(\tau_i) = \mathrm{wt}(\zeta_i) = 2^i \qquad \mathrm{wt}(\tau) = 0$$

and extending multiplicatively to monomials. We similarly define this on the subalgebras $A/\!\!/A(n)_{**}$.

Proposition

The Mahowald weight defines a filtration by comodules on $A^{\mathbb{C}}_{**}$ and the subalgebras $A/\!\!/A(n)^{\mathbb{C}}_{**}$.

kq co-operations continued

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Definition

Let $M_1(k)$ denote the subcomodule of $A/\!\!/A(1)^{\mathbb{C}}_{**}$ spanned by monomials of weight exactly $4k$.

Proposition (C.-Quigley)

There is a splitting in $A(1)^{\mathbb{C}}_{**}$-comodules

$$A/\!\!/A(1)^{\mathbb{C}}_{**} \cong \bigoplus_{i \geq 0} M_1(k)$$

Remark

We speculate that there are motivic versions of the integral Brown-Gitler spectra $H\mathbb{Z}_i^{mot}$ and that there are isomorphisms

$$H_*\Sigma^{4i,2i} H\mathbb{Z}_i^{mot} \cong M_1(k).$$

kq co-operations continued

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We have analogues of Mahowald's exact sequences

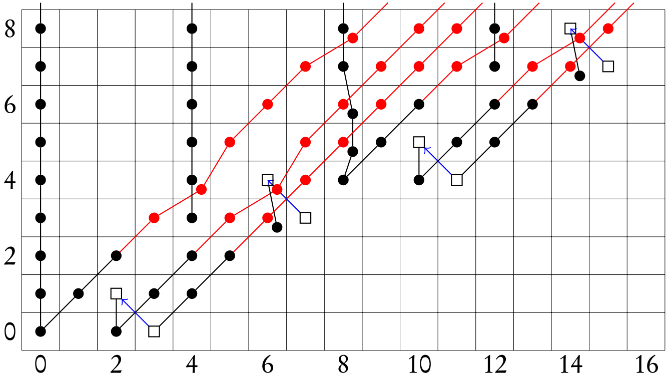
$$0 \to \Sigma^{4j,2j} H\underline{\mathbb{Z}}_j \to H\underline{\mathbb{Z}}_{2j} \to \underline{kq}_{j-1} \otimes (A(1)/\!\!/A(0))^\vee \to 0,$$

$$0 \to \Sigma^{4j,2j} H\underline{\mathbb{Z}}_j \otimes H\underline{\mathbb{Z}}_1 \to H\underline{\mathbb{Z}}_{2j+1} \to \underline{kq}_{j-1} \otimes (A(1)/\!\!/A(0))^\vee \to 0.$$

This allows us to inductively compute $\mathrm{Ext}_{A(1)^{\mathbb{C}}_{**}}(H\underline{\mathbb{Z}}_i)$.

kq co-operations continued

Here is a picture of how we compute $\mathrm{Ext}_{A(1)^{\mathbb{C}}_{**}}(H\underline{\mathbb{Z}}_1)$.



kq co-operations continued

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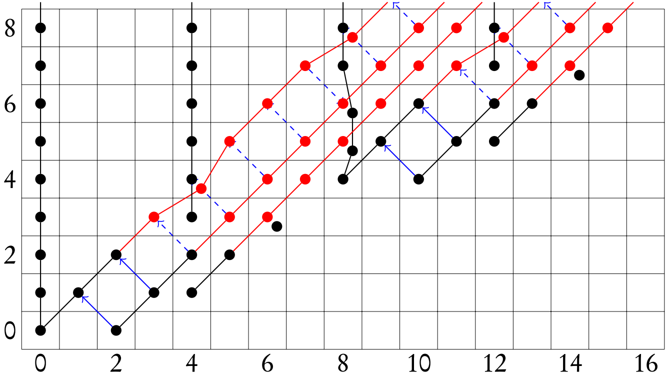
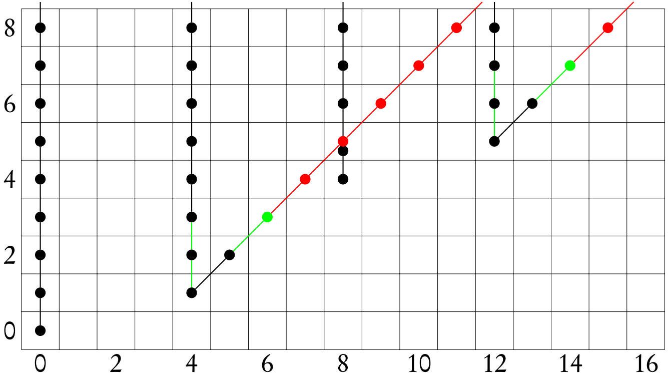


kq co-operations continued



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Differentials

Recall that there is a (strict) symmetric monoidal functor, called *Betti Realization*

$$\mathsf{SH}(\mathbb{C}) \to \mathsf{Sp}; X \mapsto X(\mathbb{C}).$$

This map is quite nice on homotopy groups: it just sets τ to 1. That is,

$$\pi_{*,*}(X) \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[\tau^{\pm 1}] \cong \pi_*(X(\mathbb{C})) \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[\tau^{\pm 1}]$$

Moreover, we get a morphism of Adams spectral sequences,

$$E_r(X; kq) \to E_r(X(\mathbb{C}), ko)$$

This allows us to deduce many differentials.

Differentials

Mahowald was able to prove the following.

Theorem (Mahowald)

In the ko-based ASS we have the following:

1. *Suppose that $x \in E_1^{0,4k}$ with $n \geq 2$ is a cycle under d_1. Suppose also that it is represented in $\pi_t(ko \wedge \overline{ko}^{\wedge n})$ by an element in $H\mathbb{F}_2$-Adams filtration $f \geq 2$. Then x is a boundary under d_1.*
2. *The d_1-differentials $d_1 : E_1^{0,4k} \cong \mathbb{Z} \to \mathbb{Z} \cong E_1^{1,4k}$ is multiplication by $2^{\rho(k)}$ where $\rho(k) = \nu_2(8k)$*

Differentials

We derive the $\mathbb{C}$-motivic analog of Mahowald's theorem via Betti Realization.

Theorem (C.-Quigley)

In the kq-based ASS we have the following:

1. *Suppose that $x \in E_1^{n,t,u}$ is a τ-torsion free class with $n \geq 2$ and that x is a cycle under d_1. Supose also that it is represented in $\pi_{t,u}(kq \wedge \overline{kq}^{\wedge n})$ by an element of $H\mathbb{F}_2$-Adams filtration $f \geq 2$. Then x is a d_1-boundary.*
2. *Each τ-torsion class in $\pi_{**}(kq \wedge \overline{kq}^{\wedge n})$ is in the image of d_n, or is mapped essentially under d_n, or $n = 0$ or $n = 1$ and the homotopy can be identified with $\pi_{**}(H\mathbb{Z}_1 \wedge kq)$.*
3. *The d_1-differentials $d_1 : E_1^{0,4k,u} \cong \mathbb{Z} \to \mathbb{Z} \cong E_1^{1,4k,u}$ is multiplication by $2^{\rho(k)}$ where $\rho(k) = \nu_2(8k)$*

Differentials

The $\mathbb{C}$-motivic differentials are lifted from the classical ko-based ASS. This relies on the following two facts.

1. Elements in the E_1-page fall into two species: those which are τ-torsion free and those which are simple τ-torsion, i.e. x so that $\tau x = 0$.
2. The classes which are simple τ-torsion are η-torsion free and are in fact η-multiples of τ-torsion free classes.

This allows us to derive everything from the classical situation.

E_∞-page

We deduce from the differentials the following theorem.

Theorem (C.-Quigley)

1. *The 0-line of the kq-resolution is given, as a $\mathbb{Z}_2[\tau, v_1^4]$-module, by*

$$E_\infty^{0,*,*} \cong \mathbb{Z}_2[\tau]\{1\} \oplus \mathbb{M}_2 \cdot \{v_1^{4k}\eta^j \mid k, j \geq 0\} / \mathbb{M}_2\{\tau v_1^{4k}\eta^j \mid j \geq 3\}.$$

2. *The 1-line of the kq-resolution is given by*

$$E_\infty^{1,*,*} \cong \bigoplus_{k \geq 0} \Sigma^{4k}\mathbb{Z}/2^{\rho(k)}[\tau] \oplus \mathbb{M}_2[h_1, v_1^4]/(h_1^3\tau).$$

Inverting η

Given an E-based ASS and an element of $a \in E_*$, one can consider the a-localized E-based ASS. Since localization is an exact functor, we still get a spectral sequence.

Question

What does the localized Adams spectral sequence converge to? If it converges at all.

Let's consider the case when $E = kq$ and $a = \eta$. In this case, convergence is not guaranteed unless there is a horizontal vanishing line, which never happens for finite complexes. However, one can still obtain convergence with sufficient vanishing lines and bounds on torsion: It needs to be guaranteed that there cannot be an infinite family of hidden extensions relating a-torsion classes.

Inverting η

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It is an easy observation from the E_1-page of the kq-based ASS that we have the following:

Proposition (C.-Quigley)

There is a vanishing line on the E_1-page of the kq-based ASS for $S^{0,0}$ which is of slope $1/3$ and through the origin. Put another way: We have $E_1^{s,t,*} = 0$ whenever $t < 4s$.

Proof.

First note that $\pi_{t,*}\overline{kq} = 0$ if $t \leq 3$. Thus the smash powers $kq \wedge \overline{kq}^{\wedge s}$ are $4s-1$-connected. Alternatively, one can use the motivic ASS for $kq \wedge \overline{kq}^{\wedge s}$. $\square$

Inverting η

This vanishing line is sufficient to infer that the η-inverted kq-ASS converges to $\eta^{-1}S$. From our calculation of the 0- and 1-lines we see that

$$\mathbb{F}_2[\eta^{\pm 1}, \mu, \epsilon]/(\epsilon^2) \subseteq \pi_{**}\eta^{-1}S^{0,0}.$$

where $|\mu| = (9,5)$ and $|\epsilon| = (8,5)$. The class μ is detected by $v_1^4\eta[0]$ on the 0-line and ϵ is detected by $2v_1^2\eta[1]$ on the 1-line. Equality follows from the following:

Proposition (Bounded η-torsion)

If $x \in E_\infty^{n,*,*}(S^{0,0}, kq)$ is nonzero and $n \geq 2$, then $\eta^2 x = 0$.

This, along with the $1/3$-vanishing lines shows that inverting η on E_∞ recovers $\eta^{-1}S^{0,0}$.

Inverting v_1

We would like to perform a similar analysis to study v_1-periodicity in the motivic sphere. This is analogous to Mahowald's program. We need the following.

Theorem (C.-Quigley)

We have a 1/5 vanishing line. More precisely, we have that $E_\infty^{n,t,u} = 0$ whenever $6n > t + 7$.

The proof of this theorem is quite difficult, even classically. Our strategy is to mimic Mahowald's approach. We prove: (1) the spectrum $A_1 := S^{0,0}/(2, \eta, v_1)$ has kq-resolution having a 1/5-vanishing line, (2) from a v_1-BSS we derive that $Y := S/(2, \eta)$ has a 1/5-vanishing line, (3) by carefully analysing the cofibrations defining Y and $S/2$ we derive that $S/2$ and S having 1/5 vanishing lines, resp.

Inverting v_1

Combining the above we arrive at

Theorem (C.-Quigley)

The only v_1-periodic elements of $S^{0,0}$ are those v_1-periodic elements in E_∞^0 and E_∞^1.

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Classical telescope conjecture

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One formulation of the classical telescope conjecture is the following: Suppose that X is a finite p-local complex of type n. By nilpotence we now that X is a v_n-self map $f : \Sigma^d X \to X$. We have the telescope

$$f^{-1}X = \operatorname{colim}\left(X \to \Sigma^{-d}X \to \Sigma^{-2d}X \to \cdots\right)$$

It turns out that $f^{-1}X$ is $K(n)$-local, and so there is a map

$$f^{-1}X \to L_{K(n)}X.$$

Conjecture (telescopic formulation, Ravenel)

The above map is an equivalence.

Classical telescope conjecture

Another formulation is given by Miller: Let L_n^f denote the localization functor which arises from localizing with respect to *finite* complexes which are acyclic with respect to $K(\leq n) := K(0) \vee \cdots \vee K(n)$. Since finite $K(\leq n)$-acyclics are also $K(\leq n)$-acyclics, we get a natural morphism $L_n^f \to L_n$. The telescope conjecture is equivalent to

Conjecture (telescope conjecture, localization formulation)

The natural map $L_n^f \to L_n$ is an equivalence.

It has been shown that both L_n^f and L_n are smashing localizations, so this conjecture is also equivalent to the following.

Conjecture (telescope conjecture, smashing formulation)

The map $L_n^f S^0 \to L_n S^0$ is an equivalence.

Motivic Telescope conjectures

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The chromatic picture motivically is far more complicated, even over $\mathbb{C}$. Recent work of Gheorghe and Krause has shown there are *exotic motivic Morava K-theories*. Gheorghe constructs $K(w_n)$ where

$$K(w_n)_{**} = \mathbb{F}_2[w_n^{\pm}] \ |w_n| = (2^{i+2}-3, 2^{i+1}-1).$$

Here, $w_0 = \eta$ and w_1 is a non-nilpotent element on $S^{0,0}/\eta$. Krause constructs even more exotic ones $K(\beta_{ij})$ for $i > j \geq 0$; he shows that $K(\beta_{n,0}) = K(w_{n-1})$. We have

$$K(\beta_{i,j})_{**} = \mathbb{F}_2[\alpha_{i,j}, \beta_{ij}^{\pm 1}]/(\alpha^2 = \beta).$$

where

$$|\beta_{ij}| = (2^{j+2}(2^i - 1), 2^{j+1}(2^i - 1)).$$

We set $K(\beta_{i,-1}) := K(i)$.

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We can order the $K(\beta_{ij})$ by slope

$$d_{ij} := \begin{cases} \frac{1}{2^{i+1}-2} & j = -1 \\ \frac{\text{motivic weight of } \beta_{ij}}{\text{topological degree of } \beta_{ij}} = \frac{2^{j+1}(2^i-1)}{2^{j+2}(2^i-1)-2} & \text{else} \end{cases}$$

Definition

A finite cellular $\mathbb{C}$-motivic spectrum X is of *type* (m,n) if its homology $H_{**}X$ is τ-torsion free and if $K(\beta_{ij})_{**}X = 0$ for $i > j \geq -1$ with $d_{ij} > d_{mn}$.

Proposition (Krause)

If X is a finite motivic spectrum of type (m,n), then there is a non-nilpotent self $f : \Sigma^d X \to X$ of slope d_{mn} which induces an isomorphism in $K(\beta_{ij})$-homology. Moreover, such maps are asymptotically the same.

Motivic Telescope conjectures

Following Miller's paper on finite localization we find

Proposition

If X is of type (m,n) with non-nilpotent (m,n)-self map $v: \Sigma^d X \to X$, then the map $X \to v^{-1}X$ is a finite $K(\beta_{mn})$-localization.

This leads to the localization formulation of a motivic telescope conjecture. Let L_{mn} denote localization with respect to

$$K(\leq \beta_{mn}) = \bigvee_{d_{ij} > d_{mn}} K(\beta_{ij})$$

Conjecture

The natural transformation $L^f_{mn} \to L_{mn}$ is an equivalence.

Motivic Telescope Conjectures

One can also consider the other formulations of the telescope conjecture in this setting. However, we do not know if they are equivalent. This would follow from the following.

Conjecture (Motivic smashing conjecture)

For each (m, n) the localization functor L_{mn} is smashing.

Finite localizations are always smashing, by a theorem of Miller. Our previous computations imply

Theorem (C.-Quigley)

There is an equivalence of motivic spectra $L^f_{10}S \simeq \eta^{-1}S$.

Motivic Telescope conjectures

On the otherhand, inverting η in the kq-based ASS leads to the $\eta^{-1}kq$-based ASS, which converges to $L_{\eta^{-1}kq}S$. Now $\eta^{-1}kq \simeq cKW$, connective Witt theory. Thus, our calculations show that the natural map

$$L_{10}^{f}S \to L_{cKW}S$$

is an equivalence. The $K(\leq \beta_{1,0})$ telescope conjecture would then follow from

Conjecture

There is an equality of Bousfield classes

$$\langle cKW \rangle = \langle K(0) \vee K(w_0) \rangle.$$

and the smashing conjecture.

Motivic Telescope Conjectures

Our study of v_1-periodicity similarly leads to some results about $v_1^{-1}S^{0,0}$.

$$v_1^{-1}S^{0,0} := \varprojlim_k v_1^{-1}S^{0,0}/(2^k)$$

Proposition

There are equivalences $L^f_{1,-1}S^{0,0} \simeq v_1^{-1}S^{0,0} \simeq L_{KQ}S^{0,0}$.

Note that $K(\leq \beta_{1,-1}) = K(0) \vee K(w_0) \vee K(1)$. On the other hand, we have that $\langle KQ \rangle = \langle KGL \vee KW \rangle$. So a $(1,-1)$-telescope conjecture would be true if the following were true.

Conjecture

We have the following equality of Bousfield classes $\langle KGL \rangle = \langle K(0) \vee K(1) \rangle$ and $\langle KW \rangle = \langle K(w_0) \vee K(1) \rangle$.

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motivic J-homomorphism

One of the original motivations for studying the ko-based ASS was to study v_1-inverted homotopy. In particular, Mahowald used it to prove the telescope conjecture (at the prime 2): the natural map

$$v_1^{-1}S \to L_{K(1)}S$$

is an equivalence. That this is true followed from the fact that the ko-based ASS has a 1/5-vanishing line.

Recall that in positive degrees, $L_{K(1)}S$ is the p-component of the image of J. In fact, $L_{K(1)}S$ is a "periodic image of J."

motivic J-homomorphism

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At the moment there is no definition/construction of a motivic J-homomorphism, though this is a matter of current study.

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Conjecture

The 0- and 1-line in the v_1-inverted kq-based ASS computes the image of J in positive stems.

motivic J-homomorphism

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Conjecture

The 0- and 1-line in the v_1-inverted kq-based ASS computes the image of J in positive stems.

What kind of person would I be if I didn't at least also propose a spectrum level definition of the image of J?

motivic J-homomorphism

In classical homotopy theory, there is a fibre sequence

$$L_{K(1)}S \longrightarrow KO \xrightarrow{\psi^3-1} KO \tag{6.2}$$

It is known that one can find a connective version of this resolution:

$$j \longrightarrow ko \xrightarrow{\psi^3-1} \Sigma^4 ksp \tag{6.3}$$

Note that $\beta^{-1}\Sigma^4 ksp = KO$. Here j denotes the connective image of J spectrum. Furthermore, upon inverting β in the ko-based ASS it becomes equivalent to (6.2).

motivic J-homomorphism

Recently, Gheorghe-Isaksen-Krause-Ricka have constructed a functor

$$\Gamma_\star : \mathsf{Sp} \to \mathsf{SH}(\mathbb{C})$$

which is lax symmetric monoidal. It takes many classical connective ring spectra to their $\mathbb{C}$-motivic analogs. For example, we have

$$\Gamma_\star(ko) \simeq kq \qquad \Gamma_\star ksp \simeq ksp^{mot}$$

Here ksp^{mot} is the very effective cover of $\Sigma^{4,2}KQ$. We also know that

$$\Gamma_\star(\Sigma^4 ksp) \simeq \Sigma^{4,2} ksp^{mot}.$$

motivic J-homomorphism

Thus, applying $\Gamma_\star$ we get a cofiber sequence

$$\Gamma_\star j \longrightarrow kq \xrightarrow{\Gamma_\star(\psi^3-1)} \Sigma^{4,2} ksp^{mot}. \tag{6.4}$$

Thus, we ask

Question

Does the kq-resolution contain a copy of (6.4)?

Furthermore, we conjecture

Conjecture

The $\mathbb{C}$-motivic spectrum $\Gamma_\star j$ corresponds to a $\mathbb{C}$-motivic connective image of J-spectrum

Thank you!
Any questions?