# A new basis for the complex $K$-theory cooperations algebra

Dominic Leon Culver

*Department of Mathematics, 255 Hurley Hall, Notre Dame, IN 46556*

**Abstract**

A classical theorem of Adams, Harris, and Switzer states that in the 0th grading of complex $K$-theory cooperations, $KU_0ku$ is isomorphic to the space of numerical polynomials. The space of numerical polynomials has a basis provided by the binomial coefficient polynomials, which gives a basis of $KU_0ku$.

In this paper, we produce a new $p$-local basis for $KU_0ku_{(p)}$ using the Adams splitting. This basis is established by using well known formulas for the Hazewinkel generators. For $p = 2$, we show that this new basis coincides with the classical basis modulo higher Adams filtration.

*Keywords:* cooperations algebra, complex K-theory, numerical polynomials, , Brown-Peterson spectrum, Hazewinkel generators

**Contents**



## 1. Introduction

The cooperations algebra $KU_*KU$ was originally computed by Adams, Harris, and Switzer in [1]. They show that $KU_*KU$ is torsion free, and hence the map

$$KU_*KU \to KU_*KU \otimes \mathbb{Q} \cong \mathbb{Q}[u^{\pm 1}, v^{\pm 1}]$$

is monic. They determine the image of this map, described in the following theorem.

**Theorem 1** (Adams-Harris-Switzer, [1]). *The map*

$$KU_*KU \to KU_*KU \otimes \mathbb{Q}$$

*gives an isomorphism between $KU_*KU$ and the ring of finite Laurent series $f(u,v)$ which satisfy the following condition: for any nonzero integers $h,k$ we have*

$$f(h\beta, k\beta) \in \mathbb{Z}[\beta^{\pm 1}, h^{-1}, k^{-1}]$$

*where $\beta$ is the Bott element.*

If we are working with the 2-local complex $K$-theory spectrum $KU$, then we can rewrite this condition as

$$KU_0KU_{(2)} \cong \{f(w) \in \mathbb{Q}[w^{\pm 1}] \mid f(k) \in \mathbb{Z}_{(2)} \text{ for all } k \in \mathbb{Z}_{(2)}^{\times}\}$$

where $w := v/u$. Since $KU$ is an even periodic ring spectrum, this determines the entire algebra $KU_*ku$. An elegant proof of this fact using a fracture square can be found in [6]. In particular, this method allows one to calculate

$$KU_0ku_{(2)} = \{g(w) \in \mathbb{Q}[w] \mid g(k) \in \mathbb{Z}_{(2)} \text{ for all } k \in \mathbb{Z}_{(2)}^{\times}\}$$

which is known as the space of *2-local semistable numerical polynomials*. This is related to the space of *2-local numerical polynomials*:

$$A := \{h(x) \in \mathbb{Q}[x] \mid h(k) \in \mathbb{Z}_{(2)} \text{ for all } k \in \mathbb{Z}_{(2)}\}$$

via the following change of coordinates

$$\mathbb{Z}_{(2)} \to \mathbb{Z}_{(2)}^{\times};\, k \mapsto 2k+1$$

A classical result is that the ring $A$ of 2-local numerical polynomials is a free $\mathbb{Z}_{(2)}$-module with basis given by the *binomial coefficient polynomials*

$$p_n(x) := \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}.$$

Via the change of coordinates above, setting $w = 2x+1$, we obtain a basis for $KU_0ku$,

$$g_n(w) = \frac{(w-1)(w-3)\cdots(w-(2n-1))}{2^n n!}.$$

At any prime $p$, another basis for $KU_0ku_{(p)}$ is discussed by Baker in [3] and [4]. In these papers, Baker gives a different basis for $KU_0ku_{(p)}$ where the role of the polynomials $p_n(x)$ are replaced by a sequence of Teichmüller characters, and he recovers a recursive formula.

When localizing at an odd prime $p$, $KU$ splits as a wedge of suspensions of the *Johnson-Wilson theory $E(1)$*. The homotopy groups of this spectrum are

$$\pi_*(E(1)) = \mathbb{Z}_{(p)}[v_1^{\pm 1}].$$

The connective cover $ku_{(p)}$ splits as a wedge of suspensions of the *truncated Brown-Peterson spectrum $BP\langle 1\rangle$*. The homotopy groups of this spectrum are

$$\pi_*(BP\langle 1\rangle) = \mathbb{Z}_{(p)}[v_1].$$

When the prime is 2, then the spectra $E(1)$ and $KU_{(2)}$ are equivalent, as are the spectra $BP\langle 1\rangle$ and $ku_{(2)}$. Using the Künneth spectral sequence, it is shown in [5] that

$$\begin{aligned} E(1)_*BP\langle 1\rangle &\simeq E(1)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} BP\langle 1\rangle_* \\ &\simeq E(1)_*[t_1, t_2, \ldots]/(\eta_R(v_2), \eta_R(v_3), \ldots) \end{aligned}$$

where the $v_i$ denote the Hazewinkel generators for $BP_*$ and $\eta_R$ denotes the right unit for the Hopf algebroid $(BP_*, BP_*BP)$. The splitting of $KU_{(p)}$ at odd primes $p$ gives a map

$$\varphi : E(1)_*BP\langle 1\rangle \to KU_*ku_{(p)} \tag{1}$$

obtained by including the summand. At the prime 2, this map is an isomorphism.

In this paper, we use the mod $p$ Adams spectral sequence for the spectrum $E(1) \wedge BP\langle 1\rangle$ to determine a basis for $E(1)_0BP\langle 1\rangle$ in terms of the generators $t_i$. Using the map $\varphi$, we find what semistable numerical polynomials these basis elements correspond to. More specifically, if we set

$$\varphi_n := \varphi\left(v_1^{-\frac{p^n-1}{p-1}} t_n\right)$$

then we determine an inductive formula for the $\varphi_n$'s. The basis for $E(1)_0BP\langle 1\rangle$ will then be the set of certain monomials on the $\varphi_n$'s. This inductive formula stems from formulas for the right unit, $\eta_R$, on the Hazewinkel generators $v_i$. Strangely, these inductive formulas bear a striking resemblance to those of Baker in [3]. The author does not know how these bases are related.

After determining a basis for $E(1)_0BP\langle 1\rangle$ at all primes, we focus on the prime 2, in which case $\varphi$ becomes an isomorphism, giving us a new basis for $KU_0ku$. We compare this new basis with the one provided by the $g_n$'s. In particular, it will be shown that the $g_n$-basis and the one produced here will be the same modulo higher Adams filtration. Our basis has the advantage that it is tightly connected to $BP$-theory and the Steenrod algebra. Moreover, our techniques furnish a basis for $E(1)_0BP\langle 1\rangle$ at odd primes, which could not be obtained before by the result of Adams-Switzer-Harris.

One of the original motivations for this paper was to develop a basis for the co-operations for the connective real $K$-theory spectrum, $ko$, which was connected to $BP$-theory. It was the hoped that that this may simplify some calculations in [8]. A natural starting point then was to first to develop this for $ku$. Unfortunately, the author has been unable to determine a new basis for $ko_*ko$ using the techniques of this paper. Another motivation of the authors was to describe a basis for $E(2)_0BP\langle 2\rangle$ in terms of $BP$-theory. However, at the moment, it appears that the formulas arising from the Hazewinkel generators are too complex to tackle $E(2)_0BP\langle 2\rangle$ with these techniques.

*Conventions*

We will write $\zeta_i$ for the conjugates of the polynomial generators $\xi_i$ in the dual Steenrod algebra. When given a prime $p$, we will write $H_*(-)$ for the functor $H_*(-;\mathbb{F}_p)$. We will write $\mathrm{Ext}_{\mathscr{A}_*}(M)$ for $\mathrm{Ext}_{\mathscr{A}_*}(\mathbb{F}_p, M)$ when $M$ is a comodule over the dual Steenrod algebra. We will also write $\mathrm{Ext}_{\mathscr{E}(1)_*}(M)$ for $\mathrm{Ext}_{\mathscr{E}(1)_*}(\mathbb{F}_p, M)$ when $M$ is a comodule over the Hopf algebra $\mathscr{E}(1)_* = E(Q_0, Q_1)_*$. If $X$ is a spectrum, we will write $M_*(X; Q_i)$ for the Margolis homology groups $M_*(H_*X; Q_i)$. We will write $P(x_1, \ldots, x_n)$ for the polynomial algebra on generators $x_1, \ldots, x_n$. When drawing Ext-charts or spectral sequences, we will always use Adams indexing.

## 2. Adams spectral sequence calculation of $E(1)_*BP\langle 1\rangle$

The goal of this section is to produce a basis for $E(1)_0BP\langle 1\rangle$ for all primes $p$ in terms of $BP$-theory. We begin by reviewing the calculation of $ku_*ku_{(2)}$ in terms of the Adams spectral sequence

$$\mathrm{Ext}_{\mathscr{A}_*}(H_*(ku \wedge ku)) \implies ku_*ku_2^{\wedge}. \tag{2}$$

The details of this calculation can be found in the last two chapters of [2]. Recall that the 2-primary dual Steenrod algebra is

$$\mathscr{A}_* = P(\zeta_1, \zeta_2, \ldots); \quad |\zeta_n| = 2^n - 1$$

and that the mod 2 homology of the spectrum $ku$

$$H_*(ku) \cong (\mathscr{A} /\!\!/ \mathscr{E}(1))_* = P(\zeta_1^2, \zeta_2^2, \zeta_3, \zeta_4, \ldots)$$

where $\mathscr{E}(1)$ denotes the subalgebra of the Steenrod algebra $\mathscr{A}$ generated by the Milnor primitives $Q_0$ and $Q_1$. Thus a change-of-rings shows that the spectral sequence is of the form

$$\mathrm{Ext}_{\mathscr{E}(1)}((\mathscr{A} /\!\!/ \mathscr{E}(1))_*) \implies ku_*ku_2^{\wedge}.$$

An important invariant needed in calculating Ext over the Hopf algebra $\mathscr{E}(1)$ is the *Margolis homology*, which we will now define.

If $X$ is a module over $\mathscr{E}(1)$, then as $\mathscr{E}(1)$ is an exterior algebra on $Q_0$ and $Q_1$, the actions by each $Q_i$ square to zero. So for each $i$, we may regard $X$ as a chain complex with differential $Q_i$. We define the *Margolis homology group with respect to $Q_i$* to be the homology of this complex, i.e.

$$M_*(X; Q_i) := \ker Q_i / \operatorname{im} Q_i.$$

An easy calculation (cf. [2]) shows that

$$M_*(ku; Q_0) \cong P(\zeta_1^2)$$

and

$$M_*(ku; Q_1) \cong E(\zeta_1^2, \zeta_2^2, \zeta_3^2, \ldots).$$

There is a *weight filtration* on $\mathscr{A}_*$ given by setting

$$\mathrm{wt}(\zeta_k) = 2^{k-1}$$

and extending to general monomials by

$$\mathrm{wt}(xy) = \mathrm{wt}(x) + \mathrm{wt}(y).$$

The weight filtration gives a decomposition of $\mathscr{E}(1)_*$-comodules.

$$(\mathscr{A} /\!/ \mathscr{E}(1))_* \cong_{\mathscr{E}(1)_*} \bigoplus_{k=0}^{\infty} M_1(k)$$

where the $M_1(k)$ denote the subspaces spanned by monomials whose weight is exactly equal to $2k$. These turn out to be subcomodules and they are related to the homology of the *integral Brown-Gitler spectra*. The Margolis homology of the subcomodules $M_1(k)$ have an interesting property.

**Proposition 1.** *The Margolis homology groups of $M_1(k)$ are the subspaces of the Margolis homology of $(\mathscr{A} /\!/ \mathscr{E}(1))_*$ spanned by the weight $2k$ monomials. In particular*

$$M_*(M_1(k); Q_0) = \mathbb{F}_2\{\zeta_1^{2k}\}$$

*and if the binary expansion of $k$ is*

$$k = k_0 + k_1 2 + k_2 2^2 + \cdots$$

*then*

$$M_*(M_1(k); Q_1) = \mathbb{F}_2\{\zeta_1^{2k_0}\zeta_2^{2k_1}\zeta_3^{2k_2}\cdots\}.$$

*In particular, the Margolis homology groups of $M_1(k)$ are one dimensional.*

Let $I$ denote the augmentation ideal of $\mathscr{E}(1)$. Since both of the Margolis homology of $M_1(k)$ is one-dimensional, it follows from Proposition 16.3 of [2] that $M_1(k)$ is an invertible $\mathscr{E}(1)$-module, and that the stable class of $M_1(k)$ is $\Sigma^a I^{\otimes b}$ for some unique integers $a, b$. The integers $a$ and $b$ are uniquely determined by the degrees of the nonzero elements in the $Q_0$- and $Q_1$-Margolis homology.

**Theorem 2.** *(Adams, [2]) There is a stable isomorphism between $M_1(k)$ and $\Sigma^{k+\alpha(k)} I^{\otimes(k-\alpha(k))}$ where $\alpha(k)$ is the number of 1's in the dyadic expansion of $k$. That is, there is an isomorphism of $\mathscr{E}(1)$-modules*

$$M_1(k) \oplus F \cong \Sigma^{k+\alpha(k)} I^{\otimes k-\alpha(k)} \oplus F'$$

*where $F, F'$ are free $\mathscr{E}(1)$-modules.*

*Proof.* The degrees of $Q_0$ and $Q_1$ are respectively 1 and 3. As pointed out right after Proposition 16.3 in [2], the integers $a$ and $b$ must satisfy

$$a + b = |\zeta_1^{2k}| = 2k$$

and

$$a + 3b = |\zeta_1^{2k_0}\zeta_2^{2k_1}\zeta_3^{2k_2}\cdots| = 4k - 2\alpha(k).$$

Subtracting the first from the second shows that

$$2b = 2k - 2\alpha(k)$$

and consequently $b = k - \alpha(k)$. It immediately follows that $a = k + \alpha(k)$, and the theorem follows. $\square$

The utility of this theorem lies in that it allows us to compute the $\mathrm{Ext}_{\mathscr{E}(1)_*}$ groups of $M_1(k)$ modulo $v_1$-torsion. Thus we have

**Corollary 1.** *There is an isomorphism of $Ext_{\mathscr{E}(1)_*}(\mathbb{F}_2)$-modules*

$$Ext_{\mathscr{E}(1)_*}(M_1(k))/tors \cong Ext_{\mathscr{A}_*}(H_*(\Sigma^{2k} ku^{\langle k-\alpha(k)\rangle}))$$

*where $ku^{\langle i\rangle}$ denotes the ith Adams cover of $ku$*[1]*.*

*Proof.* This is because the cohomology of $ku^{\langle i\rangle}$ is stably equivalent to $\Sigma^{-k+\alpha(k)} I^{\otimes k-\alpha(k)}$. Thus the cohomology of $\Sigma^{2k} ku^{\langle k-\alpha(k)\rangle}$ is stably equivalent to $\Sigma^{k+\alpha(k)} I^{\otimes k-\alpha(k)}$. The previous theorem now implies the isomorphism. $\square$

From this it follows that the Adams spectral sequence (2) collapses at $E_2$.

**Remark 3.** Here is an algorithm for determining the Ext-charts for $M_1(k)$.

1. Start with the monomial $\zeta_1^{2k}$, this is placed in bidegree $(2k, 0)$ and generates a free $\mathbb{F}_2[v_0, v_1]$-module.
2. If $2k \geq 4$ place $\zeta_1^{2k-4}\zeta_2^2$ in $(2k+2, 0)$ and put in the relation $v_1\zeta_1^{2k} = v_0\zeta_1^{2k-4}\zeta_2^2$. The monomial $\zeta_1^{2k-4}\zeta_2^2$ generates a free $\mathbb{F}_2[v_0, v_1]$-module.
3. Suppose we have carried out the algorithm $j$ times to produce a monomial $m$ in $(2k+2j, 0)$. Let $k$ be the first natural number such that the power of $\zeta_k$ in $m$ is at least 4. Then place $\zeta_k^{-4}\zeta_{k+1}^2 m$ at $(2k+2j+2, 0)$ and let it generate a free $\mathbb{F}_2[v_0, v_1]$-module. Put in the relation $v_1 m = v_0\zeta_k^{-4}\zeta_{k+1}^2 m$.
4. Keep performing the last step until one produces a monomial all of whose instances of $\zeta_k$ has a power less than 4.

This algorithm is used to produce the Adams chart of $M_1(4)$ in the example below. It can be shown that the last monomial produced in this algorithm is the nonzero element of $M_*(M_1(k); Q_1)$.

The algebra $KU_*ku_{(2)}$ is obtained from $ku_*ku_{(2)}$ by inverting the element $v_1$, thus it is the direct sum of the modules

$$v_1^{-1}\,\mathrm{Ext}_{\mathscr{E}(1)}(M_1(k)).$$

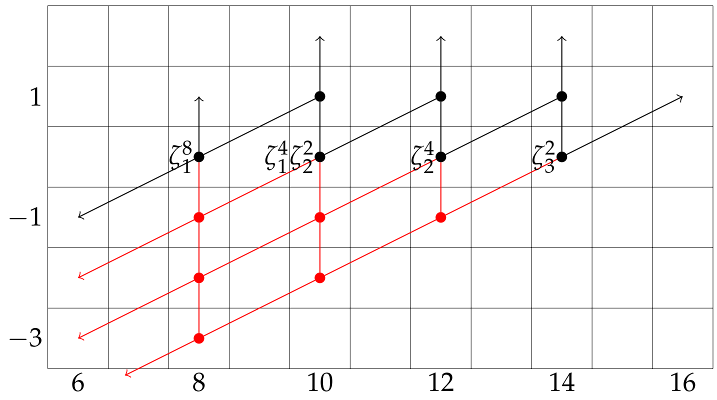
We will now calculate these $v_1$-inverted Ext-groups. Here is an example of the Adams chart for $v_1^{-1}\,\mathrm{Ext}(M_1(4))$.

**Example 1.** We will calculate $v_1^{-1}\,\mathrm{Ext}_{\mathscr{E}(1)_*}(M_1(4))$ and find a $\mathbb{Z}_{(2)}$-generator in degree 8. Here is a picture of the Adams chart.

[1]Let $X$ be a connective finite type spectrum. The Adams covers of $X$ are defined by taking a minimal Adams resolution of $X$,

$$\cdots \to X^{\langle n\rangle} \to X^{\langle n-1\rangle} \to \cdots \to X^{\langle 1\rangle} \to X.$$

The term $X^{\langle n\rangle}$ is the $n$th Adams cover of $X$.




This picture is obtained by drawing the Adams chart for $\mathrm{Ext}(M_1(4))$ according to the algorithm in 3 and then drawing $v_1^{-1}$-towers on each dot on the 0-line. From the Adams cover in $\mathrm{Ext}_{\mathscr{E}(1)_*}(M_1(4))$, we see there is the relation

$$2^3\zeta_3^2 = v_1^3\zeta_1^8$$

which in $v_1^{-1}\,\mathrm{Ext}_{\mathscr{E}(1)_*}(M_1(k))$. This shows that

$$v_1^{-1}\,\mathrm{Ext}^{s,s+8}_{\mathscr{E}(1)}(M_1(4)) = \mathbb{F}_2[v_0]\{v_1^{-3}\zeta_2^3\}.$$

Moreover, the module

$$v_1^{-1}\,\mathrm{Ext}_{\mathscr{E}(1)_*}(M_1(4))$$

is free of rank 1 generated by $\zeta_3^2$. This also shows that the contribution of $v_1^{-1}\,\mathrm{Ext}_{\mathscr{E}(1)}(M_1(4))$ to $KU_0ku$ is the free $\mathbb{Z}_{(2)}$-module generated by $v_1^{-7}\zeta_3^2$. Observe that

$$M_*(M_1(4);Q_1) = \mathbb{F}_2\{\zeta_2^3\},$$

i.e. that $\zeta_2^3$ is the unique nonzero class in the $Q_1$-Margolis homology of $M_1(4)$.

The behavior seen in the example is general. If $m_k$ denotes the nonzero element in $M_*(M_1(k);Q_1)$, then the $\mathbb{F}_2[v_0,v_1^{\pm 1}]$-module

$$v_1^{-1}\,\mathrm{Ext}_{\mathscr{E}(1)_*}(M_1(k))$$

is free of rank 1 with generator $m_k$.

**Proposition 2.** *Let $k = k_0 + k_1 2 + k_2 2^2 + \cdots$ be a natural number, then as a module over $\mathbb{F}_2[v_0, v_1^{\pm}]$, the modules $v_1^{-1}\,\mathrm{Ext}_{\mathscr{E}(1)_*}(M_1(k))$ are generated by $\zeta_1^{2k_0}\zeta_2^{2k_1}\cdots$ and the contribution of this module to $KU_0ku_{(2)}$ is a free $\mathbb{Z}_{(2)}$-module generated by $v_1^{\alpha(k)-2k}\zeta_1^{2k_0}\zeta_2^{2k_1}\cdots$.*

*Proof.* The first assertion of this proposition follows from the discussion above and the fact that

$$M_*(M_1(k);Q_1) = \mathbb{F}_2\{\zeta_1^{2k_0}\zeta_2^{2k_1}\zeta_3^{2k_2}\cdots\}.$$

The second assertion follows for degree reasons. $\square$

Recall that in the Adams spectral sequence for $BP_*BP$,

$$\mathrm{Ext}_{\mathscr{E}_*}(P(\zeta_1^2, \zeta_2^2, \zeta_3^2, \ldots)) \implies BP_*BP,$$

the elements $t_i \in BP_*BP$ are detected by $\zeta_i^2$. Since

$$\begin{aligned} E(1)_*BP\langle 1\rangle &\simeq E(1)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} BP\langle 1\rangle \\ &\simeq E(1)_*[t_1, t_2, \ldots]/(\eta_R(v_2), \eta_R(v_3), \ldots) \end{aligned}$$

the elements $\zeta_i^2$ in the Adams spectral sequence for $E(1)_*BP\langle 1\rangle$ detect $t_i$. With this notation, we conclude[2]

**Corollary 2.** *Let $\varphi_n = v_1^{-2^n+1}t_n$ in $KU_0ku_{(2)}$. The following monomials*

$$\varphi_1^{\epsilon_1}\varphi_2^{\epsilon_2}\cdots$$

*with $\epsilon_j \in \{0,1\}$ forms a basis for the free $\mathbb{Z}_{(2)}$-module $KU_0ku_{(2)}$.*

At an odd prime $p$, the dual Steenrod algebra is given by

$$\mathscr{A}_* = P(\zeta_1, \zeta_2, \ldots) \otimes E(\overline{\tau}_0, \overline{\tau}_1, \ldots)$$

where $|\zeta_n| = 2(p^n - 1)$ and $|\overline{\tau}_n| = 2p^n - 1$. The mod $p$ homology of $BP\langle 1\rangle$ is given by

$$H_*(BP\langle 1\rangle) = (\mathscr{A} /\!\!/ E(Q_0, Q_1))_*$$

where the $Q_0, Q_1$ are the Milnor primitives. Concretely, this algebra is

$$(\mathscr{A} /\!\!/ E(Q_0, Q_1))_* = P(\zeta_1, \zeta_2, \zeta_3, \ldots) \otimes E(\overline{\tau}_2, \overline{\tau}_3, \ldots).$$

There is a left action of $E(Q_0, Q_1)$ on $(\mathscr{A} /\!\!/ E(Q_0, Q_1))_*$ given by

$$\begin{aligned} Q_i(\overline{\tau}_k) &= \zeta_{k-i}^{p^i} \\ Q_i(\zeta_k) &= 0 \end{aligned}$$

for all $k$. This determines the action of the $Q_i$ since they are derivations. We show how to derive this action in Appendix A.

From this action, Margolis homology of $BP\langle 1\rangle$ is seen to be

$$\begin{aligned} M_*(BP\langle 1\rangle; Q_0) &= P(\zeta_1) \\ M_*(BP\langle 1\rangle; Q_1) &= P(\zeta_1, \zeta_2, \zeta_3, \ldots)/(\zeta_1^p, \zeta_2^p, \ldots). \end{aligned}$$

Similar to the 2-primary case, one can put a *weight filtration* on $(\mathscr{A} /\!\!/ E(Q_0, Q_1))_*$ by

$$\mathrm{wt}(\zeta_k) = \mathrm{wt}(\tau_k) = p^k.$$

[2]Note since $KU_0ku$ has no divisible summands, a set of elements of $KU_0ku_{(2)}$ is a basis if and only if it is a basis of $KU_0ku_2^\wedge$.

If we let $M_1(k)$ denote the subcomodule spanned by the monomials of weight exactly $pk$ then we get a decomposition of $\mathscr{E}(1)_*$-comodules

$$(\mathscr{A} /\!/ E(Q_0, Q_1))_* \cong \bigoplus_{k=0}^{\infty} M_1(k).$$

As in the 2-primary case, the Margolis homology of the subcomodules $M_1(k)$ are both one-dimensional. Moreover, as in the 2-primary case, if $k$ has $p$-adic expansion

$$k = k_0 + p^1 k_1 + p^2 k_2 + \cdots$$

then the Margolis homology of $M_1(k)$ is

$$M_*(M_1(k); Q_0) = \mathbb{F}_p\{\zeta_1^k\}, \quad M_*(M_1(k); Q_1) = \mathbb{F}_p\{\zeta_1^{k_0}\zeta_2^{k_1}\cdots\}.$$

As before, Proposition 16.3 of [2] implies that

$$M_1(k) \oplus F \cong \Sigma^a I^{\otimes \frac{k-\alpha_p(k)}{p-1}} \oplus F'$$

where $\alpha_p(k)$ is the sum of the digits in the $p$-adic expansion of $k$, $F$ and $F'$ are free modules, and $a$ satisfies

$$a + \frac{k - \alpha_p(k)}{p-1} = 2(p-1)k.$$

As before, this implies

$$\mathrm{Ext}_{E(Q_0,Q_1)}(M_1(k))/tors \cong \mathrm{Ext}_{\mathscr{A}_*}\left(H_*\left(\Sigma^{2(p-1)k} BP\langle 1\rangle^{\left\langle \frac{k-\alpha_p(k)}{p-1}\right\rangle}\right)\right).$$

From this it follows that the Adams spectral sequence for $BP\langle 1\rangle_* BP\langle 1\rangle$ collapses at the $E_2$-page.

Recall that the Adams spectral sequence for $BP_*BP$ at an odd prime is

$$\mathrm{Ext}_{E(\overline{\tau}_0,\overline{\tau}_1,\ldots)}(P(\zeta_1,\zeta_2,\ldots)) \implies BP_*BP$$

and in this spectral sequence the $\zeta_k$ detects $t_k \in BP_*BP$. Thus we shall write $t_k$ for $\zeta_k$. Then a proof similar to the proof of Proposition 2 shows that

**Proposition 3.** *Let the $p$-adic expansion of $k$ be given by $k = k_0 + k_1 p + k_2 p^2 + \cdots$. Then over $\mathbb{F}_p[v_0, v_1^{\pm 1}]$, the module $v_1^{-1}\,\mathrm{Ext}_{E(\tau_0,\tau_1)}(BP\langle 1\rangle)$ is generated by*

$$t_1^{k_0} t_2^{k_2} t_3^{k_3} \cdots$$

*and its contribution to $E(1)_0 BP\langle 1\rangle$ is a free $\mathbb{Z}_{(p)}$-module generated by*

$$v_1^{\frac{\alpha_p(k)-pk}{p-1}} t_1^{k_0} t_2^{k_1} t_3^{k_2} \cdots.$$

**Corollary 3.** *Let $\eta_n := v_1^{-\frac{p^n-1}{p-1}} t_n$. The $\mathbb{Z}_{(p)}$-module $E(1)_0 BP\langle 1\rangle$ is free with basis given by the monomials*

$$\eta_1^{k_1}\eta_2^{k_2}\cdots$$

*where each $k_i \in \{0, 1, \ldots, p-1\}$.*

## 3. Relationship to numerical polynomials

We will now determine the map

$$\varphi : E(1)_0 BP\langle 1\rangle \to KU_0 ku$$

in terms of numerical polynomials. Recall that the homotopy groups of the *integral* complex $K$-theory spectrum are

$$\pi_* KU = \mathbb{Z}[v^{\pm 1}]$$

and thus the rational homotopy groups are

$$\pi_*(KU_{\mathbb{Q}}) = \mathbb{Q}[v^{\pm 1}].$$

Thus we get

$$\pi_*(KU \wedge KU_{\mathbb{Q}}) = \mathbb{Q}[v^{\pm 1}, u^{\pm 1}]$$

where we let $u$ denote the Bott element coming from the right hand side $KU$. Similarly, the rational homotopy groups of $KU \wedge ku$ is given by

$$\pi_*(KU \wedge ku_{\mathbb{Q}}) = \mathbb{Q}[v^{\pm 1}, u].$$

Given a prime $p$, the rational homotopy groups of $E(1) \wedge BP\langle 1\rangle$ are given by

$$\pi_*(E(1) \wedge BP\langle 1\rangle_{\mathbb{Q}}) = \mathbb{Q}[v_1^{\pm 1}, u_1].$$

Moreover, at a prime $p$, there is a topological splitting

$$KU_{(p)} \simeq E(1) \vee \Sigma^2 E(1) \vee \cdots \vee \Sigma^{2(p-2)} E(1)$$

and the inclusion

$$E(1) \to KU$$

is given in homotopy by

$$\pi_* E(1) \to \pi_* KU_{(p)};\, v_1 \mapsto v^{p-1}.$$

Thus the morphism

$$\varphi : E(1) \wedge BP\langle 1\rangle \to KU \wedge ku_{(p)}$$

is given in rational homotopy by

$$\varphi_{\mathbb{Q}} : E(1)_* BP\langle 1\rangle_{\mathbb{Q}} \to KU_* ku_{\mathbb{Q}}\,;\, v_1 \mapsto v^{p-1},\ u_1 \mapsto u^{p-1}.$$

Let $w_1 := u_1/v_1$, then under $\varphi_{\mathbb{Q}}$, we have that

$$w_1 \mapsto w^{p-1}.$$

We will now determine the value of $\varphi$ on the monomials

$$\varphi_n := \varphi v_1^{-\frac{p^n-1}{p-1}} t_n.$$

To do this, we will need the following formula which determines the Hazewinkel generators

$$p\lambda_n = \sum_{0\le i<n} \lambda_i v_{n-i}^{p^i}$$

and the formula for the right unit on $\lambda_n$

$$\eta_R(\lambda_n) = \sum_{0\le i\le n} \lambda_i t_{n-i}^{p^i}.$$

One can find proofs of these formulas in part 2 of [2] and in [7]. Here the $\lambda_n$ is the coefficient of $x^{p^n}$ in the logarithm for the universal $p$-typical formal group law. We will show

**Theorem 4.** *The semistable polynomials $\varphi_n$ are given recursively by*

$$\varphi_1 = \frac{w^{p-1}-1}{p}$$

*and*

$$\varphi_n = \frac{w^{p^n-1} - p^{n-1}\varphi_{n-1}^p - \cdots - p\varphi_1^{p^{n-1}} - 1}{p^n}.$$

We will work out a few examples explicitly and then prove the theorem. Firstly, one has

$$p\lambda_1 = v_1$$

and so

$$\lambda_1 = \frac{v_1}{p}.$$

We will write $u_n$ for $\eta_R(v_n)$. This is justified because in $E(1)_*E(1)$, $\eta_R(v_1)$ is $u_1$. Applying $\eta_R$ gives

$$\eta_R(v_1/p) = \eta_R(\lambda_1) = t_1 + \lambda_1$$

and so

$$u_1 = \eta_R(v_1) = pt_1 + v_1$$

Thus

$$t_1 = \frac{u_1 - v_1}{p}$$

and so

$$\varphi_1 = \frac{w^{p-1}-1}{p}.$$

To get at $\varphi_2$, we need to compute $\eta_R(v_2)$. We have

$$p\lambda_2 = v_2 + \lambda_1 v_1^p$$

and so

$$v_2 = p\lambda_2 - \frac{v_1^{p+1}}{p}.$$

Applying $\eta_R$ we get

$$u_2 = p(t_2 + \lambda_1 t_1^p + \lambda_2) - \frac{u_1^{p+1}}{p}.$$

Rewriting this, we get

$$u_2 = pt_2 + v_1 t_1^p + v_2 + \frac{v_1^{p+1}}{p} - \frac{u_1^{p+1}}{p}.$$

Tensoring with $BP\langle 1\rangle_*$ produces the following relation in $E(1)_* BP\langle 1\rangle$:

$$0 = pt_2 + v_1 t_1^p + \frac{v_1^{p+1}}{p} - \frac{u_1^{p+1}}{p}$$

and hence

$$t_2 = \frac{u_1^{p+1} - v_1^{p+1}}{p^2} - \frac{v_1 t_1^p}{p}.$$

Multiplying by $v_1^{-p-1}$ gives

$$v_1^{-p-1} t_2 = \frac{w_1^{p+1} - p(v_1^{-1} t_1)^p - 1}{p^2}$$

which shows that

$$\varphi_2 = \frac{w^{p^2-1} - p\varphi_1^p - 1}{p^2}.$$

We will need the following lemma

**Lemma 1.** *In $E(1)_* BP\langle 1\rangle$ there is the following equality*

$$\lambda_n = \frac{v_1^{\frac{p^n-1}{p-1}}}{p^n}$$

*Proof.* This follows from the identity

$$p\lambda_n = \sum_{0\le i<n} \lambda_i v_{n-i}^{p^i}$$

and the fact that in $E(1)_* BP\langle 1\rangle$, $v_k = 0$ for $k > 1$. Thus

$$p\lambda_n = \lambda_{n-1} v_1^{p^{n-1}}.$$

Proceeding inductively gives the identity

$$\lambda_n = \frac{v_1^{p^{n-1}+p^{n-2}+\cdots+p+1}}{p^n} = \frac{v_1^{\frac{p^n-1}{p-1}}}{p^n}.$$

$\square$

We will prove our theorem from the following proposition.

**Proposition 4.** *In $E(1)_* BP\langle 1\rangle$, there is the relation*

$$pt_n + \sum_{1\le i\le n} \frac{v_1^{\frac{p^i-1}{p-1}} t_{n-i}^{p^i}}{p^{i-1}} = \frac{u_1^{\frac{p^n-1}{p-1}}}{p^{n-1}}.$$

*Proof.* The formula for the Hazewinkel generators is

$$p\lambda_n = v_n + \sum_{1\le i\le n-1} \lambda_i v_{n-i}^{p^i}$$

whereby

$$v_n = p\lambda_n - \sum_{1\le i\le n-1} \lambda_i v_{n-i}^{p^i}.$$

Applying $\eta_R$ then gives

$$u_n = p\sum_{0\le i\le n} \lambda_i t_{n-i}^{p^i} - \sum_{1\le i\le n-1} \left(\sum_{0\le j\le i} \lambda_j t_{i-j}^{p^j}\right) u_{n-i}^{p^i}.$$

In $E(1)_* BP\langle 1\rangle$, the $u_k$ are zero for $k > 1$. So this gives

$$p\sum_{0\le i\le n} \lambda_i t_{n-i}^{p^i} = \sum_{0\le j\le n-1} \lambda_j t_{n-1-j}^{p^j} u_1^{p^{n-1}}.$$

Using the previous lemma, we can rewrite this as

$$p\sum_{0\le i\le n} \frac{v_1^{\frac{p^i-1}{p-1}}}{p^i} t_{n-i}^{p^i} = \left(\sum_{0\le j\le n-1} \frac{v_1^{\frac{p^j-1}{p-1}}}{p^j} t_{n-1-j}^{p^j}\right) u_1^{p^{n-1}}. \tag{3}$$

We will proceed inductively, the base case being trivial to check. Suppose that we have shown the formula for $n-1$. To complete the induction, it is enough to show that

$$\sum_{0\le j\le n-1} \frac{v_1^{\frac{p^j-1}{p-1}}}{p^j} t_{n-1-j}^{p^j} = \frac{u_1^{\frac{p^{n-1}-1}{p-1}}}{p^{n-1}}.$$

Plugging in our inductive formula for $t_{n-1}$:

$$t_{n-1} = \frac{u_1^{\frac{p^{n-1}-1}{p-1}}}{p^{n-1}} - \sum_{1\le k\le n-1} \frac{v_1^{\frac{p^k-1}{p-1}}}{p^k} t_{n-1-k}^{p^k}$$

into the right hand side of equation (3) yields

$$\frac{u_1^{\frac{p^{n-1}-1}{p-1}}}{p^{n-1}} - \sum_{1\le k\le n-1} \frac{v_1^{\frac{p^k-1}{p-1}}}{p^k} t_{n-1-k}^{p^k} + \sum_{1\le k\le n-1} \frac{v_1^{\frac{p^j-1}{p-1}}}{p^j} t_{n-1-j}^{p^j} = \frac{u_1^{\frac{p^{n-1}-1}{p-1}}}{p^{n-1}}$$

which completes the proof. $\square$

We now prove the theorem

*Proof of Theorem.* By definition,

$$\varphi_n := \varphi\left(v_1^{-\frac{p^n-1}{p-1}} t_n\right).$$

Observe that

$$\frac{p^n-1}{p-1} = p^j \frac{p^{n-j}-1}{p-1} + \frac{p^j-1}{p-1}.$$

This and the proposition then show that

$$v_1^{-\frac{p^n-1}{p-1}} t_n = \frac{w_1^{\frac{p^n-1}{p-1}}}{p^n} - \sum_{0<j\leq n} \frac{v_1^{\frac{p^j-1}{p-1}} v_1^{-\frac{p^n-1}{p-1}}}{p^j} t_{n-j}^{p^j} = \frac{w_1^{\frac{p^n-1}{p-1}}}{p^n} - \sum_{0<j\leq n} \frac{(v_1^{-\frac{p^{n-j}-1}{p-1}} t_{n-j})^{p^j}}{p^j}.$$

Applying $\varphi$ now shows that $\varphi_n$ satisfies the recursive formula, by induction. □

## 4. Comparison of the $\varphi_n$ and the $g_n$

In this section we will let $p = 2$, so that $E(1)$ is equivalent to $KU_{(2)}$ and $BP\langle 1\rangle$ is equivalent to $ku_{(2)}$. Thus the map $\varphi$ is an isomorphism providing $KU_0 ku_{(2)}$ with the basis provided by the $\varphi_n$'s. In this section we compare this basis with the basis provided by the $g_n$'s. In particular we show that the bases are the same modulo higher Adams filtration.

Recall that in the Adams spectral sequence computing $\pi_* BP$:

$$\mathrm{Ext}_{\mathscr{A}_*}(H_* BP) \implies \pi_* BP_2^\wedge$$

the elements $v_i$ have Adams filtration 1. Also, in the ASS computing $BP_* BP$,

$$\mathrm{Ext}_{\mathscr{A}_*}(H_*(BP \wedge BP)) \implies \pi_*(BP \wedge BP)_2^\wedge$$

the elements detecting $t_i$ have Adams filtration 0. Moreover, the map

$$\varphi : E(1)_* BP\langle 1\rangle \to KU_* ku_{(2)}$$

preserves Adams filtration. Therefore, as $\varphi_n$ is the image of $v_1^{-2^n+1} t_n$ under $\varphi$, we can conclude:

**Proposition 5.** *The Adams filtration of $\varphi_n$ is given by*

$$\mathrm{AF}(\varphi_n) = -(2^n - 1).$$

The Adams filtration of the semistable numerical polynomial $g_n$ is given by (cf. section 2.3 of [6])

$$\mathrm{AF}(g_n) = \alpha(n) - 2n$$

where $\alpha(n)$ denotes the number of 1's in the binary expansion of $n$. We will equate the $g_n$ with products of $\varphi_n$ modulo elements of higher Adams filtration. Let $n$'s binary expansion be

$$n = n_0 + n_1 2 + n_2 2^2 + \cdots + n_\ell 2^\ell.$$

Then

$$\mathrm{AF}(\varphi_1^{n_0}\varphi_2^{n_1}\cdots\varphi_{\ell+1}^{n_\ell}) = \sum_{i=0}^{\ell} n_i(1-2^{i+1}) = \alpha(n) - 2n$$

so $g_n$ and $\varphi_1^{n_0}\varphi_2^{n_1}\cdots\varphi_{\ell+1}^{n_\ell}$ have the same Adams filtration. We will prove the following.

**Proposition 6.** *Given $n$ and its dyadic expansion*

$$n = n_0 + n_1 2 + n_2 2^2 + \cdots$$

*we have that*

$$g_n \equiv \varphi_1^{n_0}\varphi_2^{n_1}\cdots \quad \text{mod } \textit{higher Adams filtration}.$$

To prove this proposition, we will need to prove several lemmas, which is done below.

**Lemma 2.** *We have*

$$\varphi_n \equiv \frac{\varphi_1^{2^{n-1}}}{2^{2^{n-1}-1}} \quad \text{mod } \textit{higher Adams filtration}.$$

*Proof.* The map $\varphi$ preserves Adams filtration. Moreover, from Proposition 2, in $v_1^{-1}\,\mathrm{Ext}(M_1(2^{n-1}))$, there is the relation

$$2^{2^{n-1}-1} v_1^{-(2^{n-1}-1)} t_n = t_1^{2^{n-1}}.$$

Multiplying by $v_1^{-2^{n-1}}$ and applying $\varphi$ gives the desired relation. $\square$

**Lemma 3.** *We have*

$$g_n \equiv \frac{\varphi_1^n}{n!} \quad \text{mod } \textit{higher Adams filtration}.$$

*Proof.* We prove this by induction on $n$. Note that $g_1 = \varphi_1$. Suppose that we have shown that

$$g_n \equiv \frac{\varphi_1^n}{n!} \quad \text{mod higher Adams filtration.}$$

Note that

$$g_{n+1} = g_n \cdot \frac{w-(2n+1)}{2(n+1)}.$$

Even though $\frac{w-(2n+1)}{2(n+1)}$ is not an element of $KU_0 ku_{(2)}$, it is an element of $KU_0 ku \otimes \mathbb{Q}$. We will show that in $KU_0 ku \otimes \mathbb{Q}$, the element $g_{n+1}$ is congruent to $\frac{\varphi_1^{n+1}}{(n+1)!}$ modulo higher Adams filtration in $KU_0 ku \otimes \mathbb{Q}$, where Adams filtration is extended to $KU_0 ku \otimes \mathbb{Q}$ by setting

$$\mathrm{AF}\left(\frac{x}{2^i}\right) = \mathrm{AF}(x) - i.$$

This will complete the induction process because the map

$$KU_*ku \to KU_*ku \otimes \mathbb{Q}$$

preserves Adams filtration and is monic, inducing a monomorphism on associated graded spaces

$$E^0KU_*ku \to E^0KU_*ku \otimes \mathbb{Q}.$$

Note that

$$\mathrm{AF}(g_{n+1}) = \alpha(n+1) - 2n - 2$$

and also that

$$\mathrm{AF}\left(\frac{w-(2n+1)}{2(n+1)}\right) = \alpha(n+1) - \alpha(n) - 2.$$

From the formula

$$\nu_2(n!) = n - \alpha(n)$$

we find

$$\nu_2(n+1) = \nu_2((n+1)!) - \nu_2(n!) = 1 - \alpha(n+1) + \alpha(n).$$

Thus

$$\mathrm{AF}\left(\frac{w-(2n+1)}{2(n+1)}\right) = \mathrm{AF}\left(\frac{\varphi_1}{n+1}\right) = -1 - \nu_2(n+1).$$

This suggests that these numerical polynomials might be equivalent modulo higher Adams filtration. Indeed,

$$\frac{w-(2n+1)}{2(n+1)} - \frac{w-1}{2(n+1)} = \frac{-2n}{2(n+1)} = \frac{-n}{n+1}$$

and

$$\mathrm{AF}\left(\frac{n}{n+1}\right) = \nu_2(n) - \nu_2(n+1) > -1 - \nu_2(n+1)$$

and so, in $E^0(KU_*ku \otimes \mathbb{Q})$,

$$\frac{w-(2n+1)}{2(n+1)} \equiv \frac{\varphi_1}{n+1} \quad \text{mod higher Adams filtration}$$

which implies that

$$g_{n+1} \equiv \frac{\varphi_1^{n+1}}{(n+1)!} \quad \text{mod higher Adams filtration}$$

which completes the induction. □

**Corollary 4.** *We have the following congruence*

$$g_n \equiv \frac{\varphi_1^n}{2^{n-\alpha(n)}} \quad \text{mod } \textit{higher Adams filtration}.$$

*Proof.* This is because the 2-adic valuation of $n!$ is

$$\nu_2(n!) = n - \alpha(n).$$

□

**Lemma 4.** *There is the following congruence*

$$g_{2^n} \equiv \varphi_{n+1} \quad \text{mod } \textit{higher Adams filtration}.$$

*Proof.* The previous corollary gives that

$$g_{2^n} \equiv \frac{\varphi_1^{2^n}}{2^{2^n - \alpha(2^n)}} \quad \text{mod higher Adams filtration}$$

and by Lemma 2

$$\varphi_{n+1} \equiv \frac{\varphi_1^{2^n}}{2^{2^n - 1}} \quad \text{mod higher Adams fitration.}$$

Since $\alpha(2^n) = 1$, this proves the lemma. □

We can now prove Proposition 6

*Proof of Proposition 6.* First observe that if we take the binary expansion of $n$

$$n = n_0 + n_1 2 + n_2 2^2 + \cdots$$

then

$$g_n \equiv g_1^{n_0} g_2^{n_1} g_{2^2}^{n_2} \cdots \quad \text{mod higher Adams filtration} \tag{4}$$

Indeed, by Corollary 4, we have the congruence

$$g_n \equiv \frac{\varphi_1^n}{n!} \quad \text{mod higher Adams filtration}$$

and

$$g_1^{n_0} g_2^{n_1} g_{2^2}^{n_2} \cdots \equiv \left(\frac{\varphi_1}{2^0!}\right)^{n_0} \left(\frac{\varphi_1^2}{2^1!}\right)^{n_1} \left(\frac{\varphi_1^{2^2}}{2^2!}\right)^{n_2} \cdots \quad \text{mod higher Adams filtration.}$$

In this last expression, the right hand side is equal to

$$\frac{\varphi_1^n}{(2^0!)^{n_0}(2^1!)^{n_1}(2^2!)^{n_2}\cdots}.$$

So in order to show (4), it needs to be shown that

$$\nu_2(n!) = \nu_2((2^0!)^{n_0}(2^1!)^{n_1}(2^2!)^{n_2}\cdots).$$

The right hand is equal to

$$\sum_i n_i \nu_2(2^i!) = \sum_i n_i 2^i - n_i = n - \alpha(n) = \nu_2(n!)$$

and this proves the congruence (4).

To prove the proposition, apply Lemma 4 to the right hand side of (4). This gives

$$g_n \equiv \varphi_1^{n_0} \varphi_2^{n_1} \varphi_3^{n_2} \cdots$$

completing the proof of Proposition 6. □

## A. (Co)module structures at odd primes

In this section, $p$ will be a fixed odd prime and we will let $H$ denote the mod $p$ Eilenberg-MacLane spectrum $H\mathbb{F}_p$. The purpose of this section is derive the action by $E(Q_0, Q_1)$ on the mod $p$ homology of $BP\langle 1\rangle$ described in section 2. Recall that for a spectrum $X$, its mod $p$ homology is described as

$$H_*(X;\mathbb{F}_p) = \pi_*\left(H\mathbb{F}_p \wedge X\right).$$

Recall also that the Steenrod algebra is the endomorphisms of $H\mathbb{F}_p$ in the stable homotopy category, that is

$$\mathscr{A} = [H\mathbb{F}_p, H\mathbb{F}_p].$$

Thus $\mathscr{A}$ naturally acts on $H\mathbb{F}_p \wedge BP\langle 1\rangle$, i.e. if $a \in [H\mathbb{F}_p, H\mathbb{F}_p]$ then there is the map

$$H\mathbb{F}_p \wedge X \xrightarrow{a\wedge 1_X} H\mathbb{F}_p \wedge X,$$

producing a left $\mathscr{A}$-module structure on $H_*X$. On the other hand, $H_*X$ is naturally a $\mathscr{A}_*$-comodule. These two structures on $H_*X$ are compatible with each other. The following is a consequence of Proposition 17.10 of [9].

**Proposition 7.** *Let $a \in \mathscr{A}$ and let $u \in H_*X$ and let the $\mathscr{A}_*$-coaction on $u$ be*

$$\alpha(u) = \sum_i e_i \otimes u_i \in \mathscr{A}_* \otimes H_*X.$$

*Then*

$$a \cdot u = \sum_i \langle a, \chi e_i\rangle\, u_i$$

*where $\chi$ is the conguation on $\mathscr{A}_*$.*

We will apply this proposition to the homology of $BP\langle 1\rangle$. This is given by

$$H_*(BP\langle 1\rangle) = (\mathscr{A} /\!\!/ E(Q_0, Q_1))_* = P(\zeta_1, \zeta_2, \ldots) \otimes E(\overline{\tau}_2, \overline{\tau}_2, \ldots)$$

where $Q_0, Q_1$ are Milnor primitives. The coproduct map on the odd primary dual Steenrod algebra $\mathscr{A}_*$ is given by

$$\psi(\zeta_n) = \sum_{i+j=n} \zeta_j \otimes \zeta_i^{p^j}$$

and

$$\psi(\overline{\tau}_n) = 1 \otimes \overline{\tau}_n + \sum_{i+j=n} \overline{\tau}_i \otimes \zeta_j^{p^i}.$$

From these formulas, the restriction of the coproduct to $(\mathscr{A} /\!\!/ \mathscr{E}(1))_*$ satisfies

$$\psi : (\mathscr{A} /\!\!/ \mathscr{E}(1))_* \to \mathscr{A}_* \otimes (\mathscr{A} /\!\!/ \mathscr{E}(1))_*$$

making $(\mathscr{A} /\!\!/ \mathscr{E}(1))_*$ into a $\mathscr{A}_*$-comodule algebra. The proposition tells us that the action of the Milnor primitive $Q_k$ satisfies

$$Q_k \overline{\tau}_n = \sum_{i+j=n} \langle Q_k, \chi\overline{\tau}_i \rangle \, \zeta_j^{p^i}.$$

Since $\chi\overline{\tau}_i = \tau_i$, and as $\langle Q_k, \tau_i \rangle = \delta_{ik}$, we obtain

$$Q_k \overline{\tau}_n = \zeta_{n-k}^{p^k}$$

as desired. Another application of the proposition and the fact that $Q_k$ pairs to 0 with any of the $\zeta_j$, we find that $Q_k \zeta_n = 0$. Since the $Q_k$ are primitive elements of $\mathscr{A}$ it follows that they act as derivations.

## References


[1] J. F. Adams, A. S. Harris, and R. M. Switzer, *Hopf Algebras of Cooperations for Real and Complex K-Theory*, Proceedings of the London Mathematical Society **s3-23** (1971), no. 3, 385–408.

[2] J.F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, 1974.

[3] Andrew Baker, *p-adic continuous functions on rings of integers and a theorem of k. mahler*, Journal of the London Mathematical Society **s2-33** (1986), no. 3, 414–420, available at `http://jlms.oxfordjournals.org/content/s2-33/3/414.full.pdf+html`.

[4] ———, *Operations and cooperations in elliptic cohomology, part 1: Generalized modular forms and the cooperation algebra*, New York Journal of Mathematics (1995).

[5] Andrew Baker and Birgit Richter, *On the cooperation algebra of the connective Adams summand*, Tbilisi Mathematical Journal **1** (2008), 33–70.

[6] Mark Behrens, Kyle Ormsby, Nathaniel Stapleton, and Vesna Stojanoska, *On the ring of cooperations for 2-primary connective topological modular forms* (201501), available at `1501.01050`.

[7] Michiel Hazewinkel, *Formal groups and applications*, Academic Press, Inc, 1978.

[8] Mark Mahowald and Wolfgang Lellmann, *The bo-Adams spectral sequence*, Transactions of the American Mathematical Society **300** (1987), no. 2, 593–623.

[9] Robert M. Switzer, *Algebraic topology - homotopy and homology*, 1st ed., Classics in Mathematics, Springer-Verag Berlin Heidelberg, 2002.