ON THE tmf-RESOLUTION OF Z

A. BEAUDRY, M. BEHRENS, P. BHATTACHARYA, D. CULVER, AND Z. XU

Abstract. We study the tmf-based Adams spectral sequence for the type 2 spectrum Z. We establish the structure of the E_1-page of this spectral sequence, and compute the d_1-differential modulo v_2-torsion. We develop a technique for performing low dimensional calculations, and use this to compute the spectral sequence fully in stems < 40. We use this computation to prove that the $K(2)$-local Adams-Novikov spectral sequence for Z studied by the third author and Egger collapses, resulting in the computation of the homotopy groups of $Z_{K(2)}$. We discuss how these computations fit with the conjectural failure of the telescope conjecture.

Contents

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1. Introduction

In [BE16a], the third author and Egger introduced a class of 2-primary type 2 spectra $\widetilde{\mathcal{Z}}$ whose elements Z all satisfy

$$H^*(Z) \cong_{A(2)} A(2) /\!\!/ E(Q_2)$$

for the subalgebras $E(Q_2) \subset A(2)$ of A, the mod 2 Steenrod algebra. Here and throughout the paper, cohomology implicitly is taken with coefficients in $\mathbb{F}_2$. The two other key features of any $Z \in \widetilde{\mathcal{Z}}$ are

- it admits a self-map $v : \Sigma^6 Z \to Z$ inducing multiplication by v_2^1 in $K(2)_*Z$, and,
- $\mathrm{tmf} \wedge Z \simeq k(2)$.

Here, tmf is the connective spectrum of topological modular forms, $K(2)$ is height 2-Morava K-theory, and $k(2)$ is its connective cover.[1]

The importance of any Z lies in the fact that it is the height 2 analogue of the type 1 spectrum $Y := M(2) \wedge C\eta$, where $M(2)$ is the mod 2 Moore spectrum and $C\eta$ is the cofiber of $\eta \colon S^1 \to S^0$. Indeed, Y satisfies height 1 analogs of some of the properties of Z:

- $H^*(Y) \cong_{A(1)} A(1) /\!\!/ E(Q_1)$,
- it admits a v_1^1-self-map $v : \Sigma^2 Y \to Y$, and,
- $\mathrm{bo} \wedge Y \simeq k(1)$

where bo denotes the connective cover of the real K-theory spectrum KO. The above properties of Y play a crucial role in the study the bo-Adams spectral sequence (bo-ASS) for Y, which was used by Mark Mahowald to prove the telescope conjecture at chromatic height 1 at the prime 2 [Mah82]. Thus it is natural to ask if the tmf-ASS of Z can shed light on the telescope conjecture at chromatic height 2 at the prime 2, a question which so far remains unanswered for chromatic heights greater than 1.

Let us briefly recall the statement of the telescope conjecture, which is due to Ravenel [Rav84]. Fix a prime p and an integer $n \geq 0$. Recall that the homotopy groups of the height n Morava K-theory spectrum are given by

$$K(n)_* \cong \mathbb{F}_p[v_n^{\pm 1}]$$

where $|v_n| = 2(p^n - 1)$. A finite p-local spectrum X is of *type* n if n is the smallest integer such that $K(n)_*(X) \neq 0$. The periodicity theorem [DHS88] of Devinatz, Hopkins and Smith guarantees that for a type n spectrum X, there exists an integer $d > 0$ and a map

$$v \colon \Sigma^{2(p^n-1)d} X \to X$$

[1] In the context of this paper, $K(2)$ will denote the form of Morava K-theory derived from the Morava E-theory spectrum $E_2 = E(\mathbb{F}_4, \widehat{C})$ where $\widehat{C}$ is the formal group law of the supersingular elliptic curve (see Section 4.1 for more details).

such that $K(n)_*(v)$ is multiplication by v_n^d. We will call the map $v\colon \Sigma^{2d(p^n-1)}X \to X$ a v_n^d-self-map. The telescope of X is defined as the homotopy colimit

$$\widehat{X} := X \xrightarrow{v} \Sigma^{-2(p^n-1)d}X \xrightarrow{v} \Sigma^{-4(p^n-1)d}X \to \dots$$

Let $X_{K(n)}$ denotes the Bousfield localization of X with respect to $K(n)$.

Conjecture 1.0.1 (Telescope Conjecture)**.** For any type n spectrum X, the natural map

$$\widehat{X} \to X_{K(n)}$$

is an equivalence.

The goal of this paper is to adapt the bo-ASS techniques used by Mahowald to study the tmf-ASS for Z

$${}^{\mathrm{tmf}}E_1^{n,t}(Z) = \pi_t(\mathrm{tmf}^{\wedge n+1} \wedge Z) \Rightarrow \pi_{t-n}Z$$

in order to study v_2-periodicity and the telescope conjecture. We transfer the bo-ASS techniques developed in [BBB+17] to compute the tmf-ASS for Z.

Specifically, we will give a method for computing the E_2-page of the tmf-ASS for Z through a range, and use this to perform low dimensional computations of π_*Z for a particular choice of $Z \in \widetilde{\mathcal{Z}}$.[2] These low dimensional computations will allow us to eliminate the possibility of differentials in the localized tmf-ASS, studied in [BE16a], converging to $\pi_*Z_{K(2)}$. The resulting computation of $\pi_*Z_{K(2)}$ represents the first non-trivial computation of the homotopy groups of a $K(2)$-local finite complex at the prime 2.

Another aim of this paper is to repeat Mahowald's analysis and to identify where the arguments fail to resolve the telescope conjecture at $n = 2$ for the spectrum Z. Further, we will cast the arguments of Mahowald-Ravenel-Shick [MRS01] in this context and explain why one could be lead to suspect that the telescope conjecture might fail in this case.

Overview of the paper. In Section 2, we establish some notation and recall some facts about the 2-primary dual Steenrod algebra A and its subalgebras $A(n)$ and $E(n)$. We then recall how Margolis homology of an A-module M can be used to compute Ext groups of the form

$$\mathrm{Ext}^{*,*}_{E(Q_n)}(M, \mathbb{F}_2).$$

The importance of these Ext groups is that the classical ASS for $k(n)_*X$ takes the form

$${}^{ass}E_2^{s,t} = \mathrm{Ext}^{s,t}_{E(Q_n)}(H^*X, \mathbb{F}_2) \Rightarrow k(n)_{t-s}X.$$

We compute the Margolis homology of $A/\!\!/A(n)$ and $A/\!\!/E(n)$.

In Section 3 we begin our analysis of the tmf-ASS $\{{}^{\mathrm{tmf}}E_r^{n,t}(Z)\}$. Since the E_1-term is given by

$${}^{\mathrm{tmf}}E_1^{n,t}(Z) = \pi_t(\mathrm{tmf}^{\wedge n+1} \wedge Z) \cong k(2)_t(\mathrm{tmf}^{\wedge n}),$$

[2]We restrict attention to a particular $Z \in \widetilde{\mathcal{Z}}$ because the computations of Section 7 rely on computer Ext computations based on a particular Steenrod module structure on $H^*(Z)$. However, preliminary computations with the May spectral sequence suggest that the computations of that section hold for every $Z \in \widetilde{\mathcal{Z}}$.

we may use the Adams spectral sequence to calculate it. Our Margolis homology computations give the E_2-terms of these Adams spectral sequences, and we show these Adams spectral sequences collapse to give a short exact sequence of chain complexes

$$0 \to V^{*,*}(Z) \to {}^{\mathrm{tmf}}E_1^{*,*}(Z) \to \mathcal{C}^{*,*}(Z) \to 0 \tag{1.0.2}$$

where the groups $\mathcal{C}^{*,*}(Z)$ are v_2-torsion free and completely computable and the groups $V^{*,*}(Z)$ are v_2^1-torsion and essentially incomputable. We will refer to $\mathcal{C}^{*,*}(Z)$ as the *good complex* and $V^{*,*}(Z)$ as the *evil complex*. The goal is to use the short exact sequence (1.0.2) to compute ${}^{\mathrm{tmf}}E_2^{*,*}$ from $H^{*,*}(\mathcal{C}(Z))$ and $H^{*,*}(V(Z))$ (the latter which we will show is computable, despite the incomputability of $V^{*,*}(Z)$ itself).

The journey begins in Section 4, where we compute the differentials in the good complex $\mathcal{C}^{*,*}(Z)$. This is accomplished by showing that the good complex is actually isomorphic to the cobar complex of an explicit sub-Hopf algebra $\widetilde{\sigma}(2)$ of a quotient of the Morava stabilizer algebra $\Sigma(2)$.

In Section 5 we embark on the computation of

$$H^{*,*}(\mathcal{C}(Z)) \cong \mathrm{Ext}^{*,*}_{\widetilde{\sigma}(2)}(k(2)_*, k(2)_*).$$

The cohomology of the Morava stabilizer algebra $\Sigma(2)$ was computed by Ravenel [Rav77] using a modification of the May spectral sequence which we will call the *May-Ravenel spectral sequence*. In our setting, the May-Ravenel spectral sequence takes the form

$${}^{MR}E_1^{*,*,*} = H^{*,*}(E^0_*\mathcal{C}(Z)) \Rightarrow H^{*,*}(\mathcal{C}(Z)).$$

We completely compute the E_1-term of this spectral sequence (Theorem 5.4.3). It is possible there are higher differentials and extensions in this spectral sequence, but we will find that in the low dimensional range we consider, none can occur.

Having dealt with the good complex, in Section 6 we turn to the problem of computing the cohomology of the evil complex. The situation is analogous to that of the bo-ASS, for which the authors have already developed a technique which we refer to as the *agathokakological method* [BBB+17]. The key ingredients are the *algebraic agathokakological spectral sequence* (AKSS)

$$H^{*,*,*}(\mathcal{C}_{alg}(Z)) \oplus H^{*,*}(V(Z)) \Rightarrow {}^{ass}E_2^{*,*}(Z),$$

and the *dichotomy principle* (Theorem 6.2.6) which relates evil terms in the algebraic AKSS to v_2-torsion in ${}^{ass}E_2^{*,*}(Z)$. We therefore are able to recover $H^{*,*}(V(Z))$ from $H^{*,*,*}(\mathcal{C}_{alg}(Z))$ (which we completely compute) and ${}^{ass}E_2^{*,*}(Z)$ (which we compute using Bruner's Ext software [Bru93]). We end this section with a discussion of the topological AKSS,

$$H^{*,*}(\mathcal{C}(Z)) \oplus H^{*,*}(V(Z)) \Rightarrow \pi_*Z$$

which is essentially a refinement the tmf-ASS.

In Section 7, we perform low dimensional computations of the tmf-ASS (or equivalently, the topological AKSS) for Z in the range $t - n < 40$. This proceeds by first analyzing v_2-periodicity in ${}^{ass}E_2^{*,*}(Z)$ by analyzing the E_2-term of the Adams spectral sequence for the cofiber

$$\Sigma^6 Z \xrightarrow{v_2} Z \to A_2$$

where $H^*A_2 \cong A(2)$. Appendix A contains the Bruner module definition data used to compute the relevant Ext charts. We then compute the algebraic AKSS in our range. From this we extract $H^{*,*}(V(Z))$, which we input into the topological AKSS, and compute through our range. We end this section with a comparison to the computations of Bhattacharya-Egger of the $K(2)$-local Adams-Novikov spectral sequence (ANSS) for Z, and prove that their spectral sequence collapses by mapping the tmf-ASS to the $K(2)$-local ANSS (Theorem 7.5.1).

Mahowald proved the telescope conjecture for Y by showing that the E_2-term of the bo-ASS

$${}^{\mathrm{bo}}E_2^{n,t}(Y) \Rightarrow \pi_{n-t}Y$$

decomposes into a direct sum of two pieces:

(1) a summand which is v_1-torsion free and is isomorphic to $\pi_*Y_{K(1)}$ after v_1-inversion, and
(2) a summand which consists entirely of bounded v_1^2-torsion.

Mahowald then showed that infinite sequences of hidden extensions among this v_1^2-torsion cannot contribute to the homotopy of the telescope $\pi_*\widehat{Y}$ by proving that ${}^{bo}E_2^{*,*}(Y)$ has a vanishing line of slope $1/5$.

In Section 8 we discuss how the analog of this paradigm fails for the tmf-resolution. Namely, assuming there are no additional differentials or extensions in the May-Ravenel spectral sequence, and assuming a certain pattern of d_3-differetials, we show that ${}^{\mathrm{tmf}}E_4$ decomposes into a direct sum of three pieces:

(1) a summand which is v_2-torsion free, and is isomorphic to $\pi_*Z_{K(2)}$ after v_2 inversion,
(2) a summand which consists entirely of bounded v_2^2-torsion, and
(3) a summand which consists of unbounded v_2-torsion, and assembles via a conjectural sequence of hidden extensions, into an uncountable collection of *v_2-parabolas*.[3]

Our methods establish a slope $1/11$ vanishing line for ${}^{\mathrm{tmf}}E_2^{*,*}(Z)$, but we explain why one might expect to be able to improve this to a slope $1/13$ vanishing line, which would preclude infinite families of hidden extensions among the terms in summand (2) from assembling to give v_2-families in $\pi_*\widehat{Z}$. We then describe the analogs of conjectures of Mahowald-Ravenel-Shick [MRS01] which describe a hypothetical picture (*the parabola conjecture*) of $\pi_*\widehat{Z}$ which is assembled from a portion of the classes in summands (1) and (3) above, and in particular is unequal to $\pi_*Z_{K(2)}$. However, just as in [MRS01], it is totally possible for a bizarre pattern of differentials between v_2-parabolas to occur to make the telescope conjecture true.

[3]We call them v_2-parabolas because they lie on (sideways) parabolas in the $(t-n, n)$-plane.

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2. Background

2.1. **Subalgebras and subquotients of the Steenrod algebra.** Let A denote the mod 2 Steenrod algebra and let A_* be its dual. The algebra A_* is a polynomial algebra on the Milnor generators ξ_i of degree i. Letting $\zeta_i = \overline{\xi}_i$ be the conjugates, A_* can also be expressed as

$$A_* = \mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3, \ldots]$$

where the elements $\zeta_i = \overline{\xi}_i$ of degree $2^i - 1$ are dual to the elements $Q_{i-1} \in A$ defined inductively as

$$\begin{aligned} Q_0 &= \mathrm{Sq}^1, \\ Q_i &= [\mathrm{Sq}^{2^i}, Q_{i-1}], \quad i > 0. \end{aligned}$$

The coproduct on A_* is given by

$$\psi(\zeta_k) = \sum_{i+j=k} \zeta_i \otimes \zeta_j^{2^i}.$$

The elements Q_n are primitive, i.e., $\psi(Q_n) = Q_n \otimes 1 + 1 \otimes Q_n$ and satisfy $Q_n^2 = 0$.

Let $A(n)$ be the subalgebra generated by $\mathrm{Sq}^1, \ldots, \mathrm{Sq}^{2^n}$ and $E(n)$ be that generated by $Q_0, \ldots, Q_n$. Of particular interest will be the A-modules

$$A /\!\!/ A(n) \cong A \otimes_{A(n)} \mathbb{F}_2,$$

$A /\!\!/ E(n)$ and $A(n) /\!\!/ E(Q_n)$ since

$$\begin{array}{lll} H^*\mathrm{bo} \cong A /\!\!/ A(1), & H^*\mathrm{ku} \cong A /\!\!/ E(1), & H^*Y \cong A(1) /\!\!/ E(Q_1), \\ H^*\mathrm{tmf} \cong A /\!\!/ A(2), & H^*BP\langle 2\rangle \cong A /\!\!/ E(2), & H^*Z \cong A(2) /\!\!/ E(Q_2). \end{array}$$

We note that the dual of $A(n)$ and $E(n)$ are given by

$$\begin{aligned} A(n)_* &\cong A_*/(\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \ldots, \zeta_{n+1}^2, \zeta_{n+2}, \ldots) \\ E(n)_* &\cong E(\zeta_1, \ldots, \zeta_{n+1}). \end{aligned}$$

Hence, $A /\!\!/ A(n)$ and $A /\!\!/ E(n)$ have duals given by

$$\begin{aligned} (A /\!\!/ A(n))_* &\cong \mathbb{F}_2[\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \ldots, \zeta_{n+1}^2, \zeta_{n+2}, \ldots] \\ (A /\!\!/ E(n))_* &\cong \mathbb{F}_2[\zeta_1^2, \ldots, \zeta_{n+1}^2, \zeta_{n+2}, \zeta_{n+3}, \ldots]. \end{aligned}$$

2.2. **Margolis homology.** We will use information on the action of the subalgebra $E(Q_n)$ to carry out our computations. We gather some useful observations in this section. A reference for these topics is [Mar83, Part III].

Lemma 2.2.1. *Let M be a graded module over an exterior algebra $E(x) = \mathbb{F}_2[x]/x^2$ where x has degree k. Suppose that M is of finite type (that is, a finite $\mathbb{F}_2$-vector space in each degree). Then M is a direct sum of free modules and trivial modules.*

Definition 2.2.2. Let M be a module over $E(x)$. Let $\ker_x(M)$ be the kernel of multiplication by x and $\mathrm{im}_x(M)$ be its image. Define

$$H(M;x) := \ker_x(M)/\,\mathrm{im}_x(M).$$

Corollary 2.2.3. *Let M be a module of finite type over an exterior algebra $E(x)$ where x has degree k, then*

$$\mathrm{Ext}^{*,*}_{E(x)}(M,\mathbb{F}_2) \cong (\mathbb{F}_2[y] \otimes H(M;x)) \oplus V$$

for y in $\mathrm{Ext}^{1,k}$ and V a direct sum of copies of $\mathbb{F}_2$ in cohomological degree zero.

Proof. Since M is of finite type, it can be expressed as a direct sum

$$M \cong \bigoplus_{i\in T} \Sigma^i \mathbb{F}_2 \oplus \bigoplus_{j\in F} \Sigma^j E(x)$$

for sets T and F. Then

$$\mathrm{Ext}^{*,*}_{E(x)}(M,\mathbb{F}_2) \cong \left(\bigoplus_{i\in T} \Sigma^i \mathbb{F}_2[y]\right) \oplus \bigoplus_{j\in F} \Sigma^j \mathbb{F}_2$$

However, $H(M;x) \cong \bigoplus_{i\in T} \Sigma^i \mathbb{F}_2$, so the claim holds. □

We will apply these results to the exterior algebra $E(Q_n)$.

Definition 2.2.4. Let M be an $A(n)$-module. The *n-th Margolis homology* of M is $H(M;Q_n)$. If $M = H^*(X)$, then we abbreviate $H(H^*(X);Q_n)$ as $H(X;Q_n)$.

It follows that for any A-module M, there is an isomorphism

$$\mathrm{Ext}^{*,*}_{E(Q_n)}(M,\mathbb{F}_2) \cong (\mathbb{F}_2[v_n] \otimes H(M;Q_n)) \oplus V$$

for v_n in $\mathrm{Ext}^{1,2^{n+1}-1}$ and V a direct sum of copies of $\mathbb{F}_2$.

Let M and N be $A(n)$-modules. For an element $\alpha \in A(n)$ let

$$\psi(\alpha) = \sum \alpha_i \otimes \alpha_j$$

denote its coproduct. Then $M \otimes N$ can be given the A-module structure

$$\alpha(a \otimes b) = \psi(\alpha)(a \otimes b) = \sum \alpha_i a \otimes \alpha_j b.$$

Since Q_n is a primitive, we have $Q_n(a \otimes b) = Q_n(a) \otimes b + a \otimes Q_n(b)$. From this, one can deduce the following lemma.

Lemma 2.2.5. *Let M and N be $A(n)$-modules of finite type. Then*

$$H(M \otimes N; Q_n) \cong H(M;Q_n) \otimes H(N;Q_n).$$

Corollary 2.2.6. *If M is an $A(n)$-module of finite type, then*

$$\mathrm{Ext}^{s,t}_{E(Q_n)}(M^{\otimes k},\mathbb{F}_2) \cong \mathbb{F}_2[v_n] \otimes H(M;Q_n)^{\otimes k} \oplus V$$

where V a direct sum of copies of $\mathbb{F}_2$.

Remark 2.2.7. To compute $\mathrm{Ext}_{E(Q_n)}$, it is useful to understand the left action of Q_n on A, or equivalently, its dual right action on A_*. Indeed, if M is a left $E(Q_n)$-module and M_* is its $\mathbb{F}_2$-dual, there is a right action of $E(Q_n)$ on M_* given by $(\varphi Q_n)(a) = \varphi(Q_n(a))$ and

$$\mathrm{Ext}_{E(Q_n)}(M, \mathbb{F}_2) \cong \mathrm{Ext}_{E(Q_n)}(\mathbb{F}_2, M_*).$$

The action of $E(Q_n)$ on A_* is determined the formula

$$\langle xQ_n, \alpha \rangle = \langle x, Q_n \alpha \rangle$$

for $x \in A_*$ and $\alpha \in A$, which is straightforward to compute on the ζ_k's: since Q_n is dual to ζ_{n+1}, if $k \geq n+1$ we have

$$\langle \zeta_k Q_n, \alpha \rangle = \langle \zeta_k, Q_n \alpha \rangle = \langle \psi(\zeta_k), Q_n \otimes \alpha \rangle = \left\langle \zeta_{k-n-1}^{2^{n+1}}, \alpha \right\rangle.$$

So,

$$\zeta_k Q_n = \begin{cases} \zeta_{k-n-1}^{2^{n+1}} & k \geq n+1, \\ 0 & k < n+1. \end{cases}$$

The following result is a straightforward consequence of Remark 2.2.7 and the fact that the action of Q_n is a derivation, so that $\zeta_k^2 Q_n = 0$ for all k.

Lemma 2.2.8. *There are isomorphisms*

$$H((A/\!\!/A(n))_*; Q_n) \cong \mathbb{F}_2[\zeta_2^{2^n}, \zeta_3^{2^{n-1}}, \ldots, \zeta_{n+1}^2, \zeta_{n+2}^2, \zeta_{n+3}^2, \ldots]/(\zeta_2^{2^{n+1}}, \zeta_3^{2^{n+1}}, \ldots)$$

and

$$H((A/\!\!/E(n))_*; Q_n) \cong \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \ldots]/(\zeta_1^{2^{n+1}}, \zeta_2^{2^{n+1}}, \ldots).$$

3. The good/evil decomposition of the E_1-term

3.1. The computation of the E_1-term of the tmf-ASS for Z.

We study the tmf-ASS for Z, which has the form

$${}^{\mathrm{tmf}}E_1^{n,t} = \pi_t(\mathrm{tmf}^{\wedge n+1} \wedge Z) \Longrightarrow \pi_{t-n}(Z).$$

To understand the E_1-term, we use the classical ASS,

$$\mathrm{Ext}_A^{s,t}(H^*(\mathrm{tmf}^{\wedge n+1} \wedge Z), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathrm{tmf}^{\wedge n+1} \wedge Z) = {}^{\mathrm{tmf}}E_1^{n,t-s} \tag{3.1.1}$$

Since the cohomology of the spectrum Z satisfies

$$H^*(Z) \cong A(2)/\!\!/E(Q_2),$$

we have

$$H^*(\mathrm{tmf}^{\wedge n+1} \wedge Z) \cong A/\!\!/A(2)^{\otimes n+1} \otimes A(2)/\!\!/E(Q_2).$$

Via two change of rings isomorphisms, we get

$$\mathrm{Ext}_A^{*,*}(H^*(\mathrm{tmf}^{\wedge n+1} \wedge Z), \mathbb{F}_2) \cong \mathrm{Ext}_{E(Q_2)}^{*,*}(A/\!\!/A(2)^{\otimes n}, \mathbb{F}_2). \tag{3.1.2}$$

Let

$$\mathcal{C}_{alg}^{n,*,*}(Z) := \mathbb{F}_2[v_2] \otimes H(A/\!\!/A(2)_*, Q_2)^{\otimes n}$$

Then Corollary 2.2.3, Corollary 2.2.6, and Lemma 2.2.8 imply the following.

Proposition 3.1.3. *There is an isomorphism of $\mathbb{F}_2[v_2]$-modules*

$$\mathrm{Ext}_A^{*,*}(H^*(\mathrm{tmf}^{\wedge n+1} \wedge Z), \mathbb{F}_2) \cong \mathcal{C}_{alg}^{n,*,*}(Z) \oplus V_{alg}^{n,*,*}(Z) \tag{3.1.4}$$

where

$$\mathcal{C}_{alg}^{n,*,*} \cong \mathbb{F}_2[v_2] \otimes \left[\mathbb{F}_2[\zeta_2^4, \zeta_3^2, \zeta_4^2, \cdots]/(\zeta_2^8, \zeta_3^8, \cdots)\right]^{\otimes n}$$

and $V_{alg}^{n,*,*}(Z)$ is a direct sum of shifted copies of $\mathbb{F}_2$'s which are simple v_2-torsion (i.e., $v_2 \cdot x = 0$ for all elements x) which are concentrated in Adams filtration zero:

$$V^{n,*}(Z) := V_{alg}^{n,0,*}(Z) = V_{alg}^{n,*,*}(Z).$$

We therefore deduce:

Corollary 3.1.5. *We have an isomorphism of $\mathbb{F}_2$-vector spaces*

$$^{\mathrm{tmf}}E_1^{n,*} = \pi_*(\mathrm{tmf} \wedge \mathrm{tmf}^{\wedge n} \wedge Z) \cong V^{n,*}(Z) \oplus \mathcal{C}^{n,*}(Z), \tag{3.1.6}$$

where $V^{*,*}(Z)$ is the module defined in Proposition 3.1.3, and,

$$\mathcal{C}^{n,*}(Z) \cong \mathbb{F}_2[v_2] \otimes \left[\mathbb{F}_2[\zeta_2^4, \zeta_3^2, \zeta_4^2, \cdots]/(\zeta_2^8, \zeta_3^8, \cdots)\right]^{\otimes n}.$$

Proof. The differentials in (3.1.1) are v_2-linear as Z has a v_2-self map. The elements of $\mathcal{C}_{alg}^{n,*,*}(Z)$ are concentrated in even degrees so there can be no differentials between them. It then follows that there can be no non-zero differentials supported by elements of $V^{n,*}(Z)$ as these are v_2-torsion so cannot hit v_2-free classes. Therefore, the ASS for $\pi_*(\mathrm{tmf}^{\wedge n+1} \wedge Z)$ collapses. Since $\mathrm{tmf} \wedge Z \simeq k(2)$, there are no possible additive extensions. $\square$

Remark 3.1.7. The Adams spectral sequence argument above is not sufficient to deduce that the isomorphism (3.1.6) is an isomorphism of $k(2)_*$-modules, because in principle there could be hidden v_2 extensions on the generators of $V^{n,*}(Z)$. However, the possibility of such v_2 extensions will be ruled out in the next section.

3.2. **A topological lift of the splitting.** We now prove that the splitting (3.1.6) lifts to the category of spectra.

Lemma 3.2.1. *For all n, there is a generalized Eilenberg-MacLane spectrum HV^n and a spectrum C^n such that*

$$\mathrm{tmf}^{\wedge n+1} \wedge Z \simeq HV^n \vee C^n$$

and which recovers the splitting (3.1.6) on the level of homotopy groups.

Proof. Let X denote $\mathrm{tmf}^{\wedge n+1} \wedge Z$. As discussed above, a change-of-rings isomorphism allows us to identify the E_2-term of the ASS for X as

$$\mathrm{Ext}_A^{*,*}(H^*(\mathrm{tmf}^{\wedge n+1} \wedge Z), \mathbb{F}_2) \cong \mathrm{Ext}_{E(Q_2)}^{*,*}(A/\!\!/A(2)^{\otimes n}, \mathbb{F}_2)$$

From Lemma 2.2.1, there is a decomposition

$$A/\!\!/A(2)^{\otimes n+1} \cong_{E(Q_2)} F \oplus G$$

where F is a free $E(Q_2)$-module and G is a direct sum of trivial $E(Q_2)$-modules. In applying the change-of-rings isomorphism, one uses the sheering isomorphism

$$A/\!\!/E(Q_2) \otimes A/\!\!/A(2)^{\otimes n} \cong_A A \otimes_{E(Q_2)} (A/\!\!/A(2)^{\otimes n})$$

and hence we have that

$$H^*(X) \cong_A (A \otimes_{E(Q_2)} F) \oplus (A \otimes_{E(Q_2)} G).$$

In particular, $A \otimes_{E(Q_2)} F$ is a free A-module and $A \otimes_{E(Q_2)} G$ is a free $A/\!/E(Q_2)$-module.

Let HV^n denotes the generalized Eilenberg-MacLane spectrum whose homotopy groups are $V^{n,*}(Z)$. Then

$$H^*(HV^n) \cong A \otimes_{E(Q_2)} F.$$

Consider the Adams spectral sequence

$$\mathrm{Ext}_A^{s,t}(H^*(HV^n), H^*(X)) \implies [\Sigma^{t-s} X, HV^n] = [X, HV^n]_{t-s}.$$

Since $H^*(HV^n)$ is a free A-module, the E_2-term is concentrated in $\mathrm{Ext}_A^{0,*}$, and so the spectral sequence collapses. Thus, there is a map of spectra

$$X \to HV^n$$

which is detected by the inclusion in cohomology

$$H^*(HV^n) \cong (A \otimes_{E(Q_2)} F) \to H^*X.$$

Let C^n denote the fiber of this map, so that we have a fiber sequence

$$C^n \to X \to HV^n. \tag{3.2.2}$$

We will show there is a map $HV^n \to X$ which splits this fiber sequence. Towards this end, consider the Adams spectral sequence

$$\mathrm{Ext}_A^{s,t}(H^*X, H^*(HV^n)) \implies [HV^n, X]_{t-s}. \tag{3.2.3}$$

Applying a change-of-rings isomorphism, the E_2-page becomes

$$\mathrm{Ext}^{*,*}_{E(Q_2)}(A/\!/A(2)^{\otimes n}, A \otimes_{E(Q_2)} F). \tag{3.2.4}$$

Recall that a free $E(Q_2)$-module is also injective (cf. [Mar83, p.245]). Since $A \otimes_{E(Q_2)} F$ is a free A-module, and since A is free as an $E(Q_2)$-module, it follows that (3.2.4) is concentrated in $\mathrm{Ext}^{0,*}_{E(Q_2)}$. So the spectral sequence (3.2.3) collapses at E_2, and there is a map of spectra

$$HV^n \to X$$

which is detected by the projection map

$$H^*X \to H^*HV^n.$$

Thus we have produced maps

$$HV^n \to X \to HV^n$$

which induce

$$A \otimes_{E(Q_2)} F \leftarrow (A \otimes_{E(Q_2)} F) \oplus (A \otimes_{E(Q_2)} G) \leftarrow A \otimes_{E(Q_2)} F$$

on cohomology. It follows that the fiber sequence (3.2.2) is split and that it recovers the splitting of (3.1.6). □

Corollary 3.2.5. *The isomorphism (3.1.6) is an isomorphism of $k(2)_*$-modules.*

3.3. **The good and evil complexes.** We now upgrade the decomposition of Corollary 3.1.5 to a short exact sequence of chain complexes. The first observation is the following.

Proposition 3.3.1. *The subspace $V^{*,*}(Z)$ forms a subcomplex of ${}^{\mathrm{tmf}}E_1^{*,*}(Z)$.*

Proof. This follows from the fact that $V^{*,*}(Z)$ consists of v_2-torsion, and the differentials commute with v_2-multiplication. Differentials supported by v_2-torsion classes cannot hit v_2-torsion-free classes. □

We will call $(V^{*,*}(Z), d_1)$ the *evil complex*. Since $(V^{*,*}(Z), d_1)$ forms a sub-complex of ${}^{\mathrm{tmf}}E_1^{*,*}(Z)$, we can define $\mathcal{C}^{*,*}(Z)$ to be the quotient complex

$$0 \to V^{*,*}(Z) \to {}^{\mathrm{tmf}}E_1^{*,*}(Z) \to \mathcal{C}^{*,*}(Z) \to 0.$$

We will call $(\mathcal{C}^{*,*}(Z), d_1)$ the *good complex*.

Abbreviate $H^{*,*}(V) = H(V^{*,*}(Z), d_1)$ and $H^{*,*}(\mathcal{C}) = H(\mathcal{C}^{*,*}(Z), d_1)$. There is a long exact sequence

$$\cdots \to H^{*,*}(V) \to {}^{\mathrm{tmf}}E_2^{*,*}(Z) \to H^{*,*}(\mathcal{C}) \xrightarrow{\partial} H^{*+1,*}(V) \to \cdots . \tag{3.3.2}$$

We will see that $H^{*,*}(\mathcal{C})$ can be computed completely, while $H^{*,*}(V)$ is mysterious. We call the elements of $H^{*,*}(V)$ *evil* and those of $H^{*,*}(\mathcal{C})$ *good*.

In [BBB+17], we establish a method for computing $H^{*,*}(V)$ in a range. The idea is to use the tmf-Mahowald spectral sequence (MSS),

$${}^{\mathrm{tmf}}_{alg}E_1^{n,s,t} = \mathrm{Ext}_A^{s,t}(H^*(\mathrm{tmf}^{\wedge n+1} \wedge Z), \mathbb{F}_2) \Rightarrow \mathrm{Ext}_A^{s+n,t}(H^*(Z), \mathbb{F}_2). \tag{3.3.3}$$

with

$$d_r : {}^{\mathrm{tmf}}_{alg}E_r^{n,s,t} \to {}^{\mathrm{tmf}}_{alg}E_r^{n+r,s-r+1,t}.$$

The construction of this spectral sequence is identical to that of [BBB+17]. The E_1-term fits into an exact sequence of chain complexes

$$0 \to V_{alg}^{*,*,*}(Z) \to {}^{\mathrm{tmf}}_{alg}E_1^{*,*,*} \to \mathcal{C}_{alg}^{*,*,*}(Z) \to 0$$

(see (3.1.4)) from which we obtain a long exact sequence

$$\cdots \to H^{*,*,*}(V_{alg}) \to {}^{\mathrm{tmf}}_{alg}E_2^{*,*,*}(Z) \to H^{*,*,*}(\mathcal{C}_{alg}) \xrightarrow{\partial_{alg}} H^{*+1,*,*}(V_{alg}) \to \cdots . \tag{3.3.4}$$

We will compute the homology $H^{*,*,*}(\mathcal{C}_{alg})$ explicitly, and the abutment of this spectral sequence can be computed through a range, for example using Bruner's program. From this, we can inductively deduce information about $H^{*,*,*}(V_{alg})$, at least through a range. Further, $H^{n,s,t}(V_{alg})$ is concentrated in degree $s = 0$ and the identification of cochain complexes

$$V^{n,t}(Z) \cong V_{alg}^{n,0,t}(Z)$$

implies that

$$H^{*,*}(V) \cong H^{*,0,*}(V_{alg}).$$

This isomorphism allows us to transfer information from the tmf-MSS to the tmf-ASS.

In order to understand ${}^{\mathrm{tmf}}E_2^{*,*}(Z)$ and ${}^{\mathrm{tmf}}_{alg}E_2^{*,*,*}(Z)$, the first step is to compute $H^{*,*}(\mathcal{C})$ and $H^{*,*,*}(\mathcal{C}_{alg})$ (see Theorem 5.4.3 and Remarks 5.4.4 and 5.4.1).

4. Computation of the differentials in the good complex

Let $\Sigma(2)$ be the Hopf algebra over $K(2)_*$, given as

$$\begin{aligned}\Sigma(2) &= K(2)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(2)_* \\ &\cong \mathbb{F}_2[v_2^{\pm 1}][t_1, t_2, \ldots]/(t_k^4 - v_2^{2^k-1}t_k).\end{aligned}$$

Note that $\Sigma(2)$ is the Morava stabilizer algebra, described in [Rav86, Chapter 6]. Let $K_2 \cong E_2/\mathfrak{m}$ be the extension of $K(2)$ described in Section 2 of [GHMR05], so that

$$(K_2)_* \cong \mathbb{F}_4[u^{\pm 1}]$$

with $|u| = -2$ and

$$v_2 = u^{-3}.$$

We let

$$\Sigma_2 := (K_2)_* \otimes_{K(2)_*} \Sigma(2)$$

denote the associated Hopf algebra over $(K_2)_*$.

We will begin this section with a discussion of the extended Morava stabilizer group associated to the unique supersingular elliptic curve defined over $\mathbb{F}_2$, and its relationship with both TMF and the more traditionally studied Morava stabilizer group associated to the Honda height 2 formal group. We will then introduce a certain quotient $\overline{\Sigma}_2$ of Σ_2 associated to an open subgroup of this extended Morava stabilizer group. The main result of this section (Theorem 4.4.5) is that there is a sub-Hopf algebra

$$(k(2)_*, \widetilde{\sigma}(2)) \subset ((K_2)_*, \overline{\Sigma}_2)$$

such that the good complex is isomorphic to the associated cobar complex [Rav86, Definition A1.2.11]:

$$\mathcal{C}^{*,*}(Z) \cong C^*_{\widetilde{\sigma}(2)}(k(2)_*).$$

The cohomology of Σ_2 was essentially studied by Ravenel in [Rav86, Chapter 6], and Ravenel's approach to this computation will be used in the following section to give an essential foothold in the computation of the cohomology of the good complex.

4.1. The elliptic Morava stabilizer group and Morava stabilizer algebra.

We first recall some facts about the automorphism group of the unique supersingular elliptic curve over $\mathbb{F}_4$, and its associated formal group. We refer to [Bea17] and [Hen18] for more details in this context.

Over $\mathbb{F}_4$, the endomorphism ring of the elliptic curve $C : y^2 + y = x^3$ is the maximal order (the Hurwitz integers)

$$\mathrm{End}(C) = \mathbb{Z}\left\{1, i, j, \frac{1+i+j+k}{2}\right\}$$

in the quaternion algebra

$$D = \mathbb{Q}\langle i, j\rangle/(i^2 = -1,\ j^2 = -1,\ ij = -ji).$$

with $k := ij$ [Deu41, pp. 237-9]. Define

$$\omega = -\frac{1}{2}(1 + i + j + k).$$

Then we have

$$\omega^3 = 1, \quad \omega^2 + \omega + 1 = 0,$$

and

$$\omega i \omega^2 = j,\ \omega j \omega^2 = k,\ \omega k \omega^2 = i.$$

The automorphism group of C is the subgroup of $D^\times$ generated by

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

and ω, so we have

$$G_{24} := \mathrm{Aut}(C) = Q_8 \rtimes C_3.$$

To make this identification explicit, we may define the generators i and ω of

We define

$$T := j - k \in \mathrm{End}(C)$$

so we have

$$T^2 = -2.$$

Then D has the alternative presentation as

$$\mathbb{Q}(\omega)\langle T\rangle/(Ta = a^\sigma T,\ T^2 = -2) \tag{4.1.1}$$

where $\omega^\sigma = \omega^2$ is the action of the Galois group

$$\mathrm{Gal} := \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) = \langle \sigma \rangle.$$

For example, $i \in D$ can be expressed as $\frac{1}{1+2\omega}(1 - T)$ in (4.1.1).

Since the curve C is defined over $\mathbb{F}_2$, the Galois group Gal also acts on End(C), and hence on $\mathrm{Aut}(C)$ and D. This action is encoded in the following lemma.

Lemma 4.1.2. *The Galois action on an element $x \in D$ is given by*

$$x^\sigma = -\frac{1}{2} T x T.$$

Proof. The $\mathbb{F}_4$ points of

$$C : y^2 + y = x^3$$

form a group isomorphic to $\mathbb{F}_3 \times \mathbb{F}_3$. A basis for this $\mathbb{F}_3$-vector space is given in (x, y) coordinates by

$$P_1 := (0, 0),$$
$$P_2 := (1, \omega).$$

The generators i and ω of the group $G_{24} = \mathrm{Aut}(C)$ correspond to the automorphisms

$$i : (x, y) \mapsto (x + 1, y + x + \omega^2),$$
$$\omega : (x, y) \mapsto (\omega^2 x, y).$$

The induced action of these automorphisms on the $\mathbb{F}_4$-points of the curve C, with respect to the basis (P_1, P_2), induces a representation

$$\rho : G_{24} \hookrightarrow GL_2(\mathbb{F}_3)$$

with

$$\rho(i) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$
$$\rho(\omega) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

The Galois action on $C(\mathbb{F}_4)$ extends the representation ρ to an isomorphism

$$\widetilde{\rho} : G_{48} := G_{24} \rtimes \mathrm{Gal} \xrightarrow{\cong} GL_2(\mathbb{F}_3) \tag{4.1.3}$$

given by

$$\widetilde{\rho}(\sigma) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

One can therefore use this isomorphism to deduce that

$$i^\sigma = -i,$$
$$\omega^\sigma = \omega^2.$$

One easily checks from this that

$$T^\sigma = T$$

and the result follows from the presentation (4.1.1). ☐

The formal group of $\widehat{C}$ has -2-series

$$[-2]_{\widehat{C}}(x) = x^4.$$

The endomorphism ring of the formal group $\widehat{C}$ is the maximal order

$$\mathrm{End}(\widehat{C}) = \mathbb{W}(\mathbb{F}_4)\langle T \rangle / (Ta = a^\sigma T,\ T^2 = -2)$$

in the 2-adic division algebra

$$D_2 := D \otimes \mathbb{Q}_2,$$

where $\mathbb{W}(\mathbb{F}_4) = \mathbb{Z}_2[\omega]/(\omega^2 + \omega + 1)$ is the Witt ring. The second Morava stabilizer group

$$\mathbb{S}_2 := \mathrm{Aut}(\widehat{C})$$

is the group of units in the order $\mathrm{End}(\widehat{C})$. Since $\widehat{C}$ is defined over $\mathbb{F}_2$, its automorphism group $\mathbb{S}_2$ also gets an action of Gal, with Galois action given by

$$g^\sigma = -\frac{1}{2} T g T,$$

and we let

$$\mathbb{G}_2 := \mathbb{S}_2 \rtimes \mathrm{Gal}$$

denote the resulting extended Morava stabilizer group. The subgroup G_{48} is a maximal finite subgroup of $\mathbb{G}_2$.

It is more traditional in chromatic computations to use the Honda height 2 formal group law H over $\mathbb{F}_4$. This is the 2-typical formal group law with 2-series

$$[2]_H(x) = x^4.$$

Its endomorphism ring, given by

$$\mathbb{W}(\mathbb{F}_4)\langle S\rangle/(Sa = a^\sigma S,\ S^2 = 2)$$

is observed in [Bea17] and [Hen18] to be isomorphic to $\mathrm{End}(\widehat{C})$ by

$$T \mapsto \alpha S, \tag{4.1.4}$$

where

$$\alpha = \frac{1-2\omega}{\sqrt{-7}} \in \mathbb{W}(\mathbb{F}_4)$$

(for a choice of $\sqrt{-7} \in \mathbb{Z}_2$). The essential property of α is that

$$\alpha\alpha^\sigma = -1.$$

We will identify $\mathrm{Aut}(H)$ and $\mathrm{Aut}(\widehat{C}) = \mathbb{S}_2$ by *defining*

$$S := \alpha^{-1}T \in \mathbb{S}_2.$$

However, the action of Gal *on* $\mathbb{S}_2$ *induced from the Honda formal group is different from the action of* Gal *induced from* $\widehat{C}$*!*

We shall denote this different Galois action (associated to the Honda formal group) σ'. It is given by

$$g^{\sigma'} = \frac{1}{2}SgS.$$

We also denote this different Galois group Gal', and the corresponding extended Morava stabilizer group by

$$\mathbb{G}_2' := \mathbb{S}_2 \rtimes \mathrm{Gal}'.$$

Lemma 4.1.5. *For* $g \in \mathbb{S}_2$ *we have*

$$g^\sigma = -\alpha g^{\sigma'}\alpha^\sigma.$$

Proof. We compute

$$\begin{aligned} g^\sigma &= -\frac{1}{2}TgT \\ &= -\frac{1}{2}\alpha Sg\alpha S \\ &= -\alpha\frac{1}{2}SgS\alpha^\sigma \\ &= -\alpha g^{\sigma'}\alpha^\sigma. \end{aligned}$$

□

Every element $g \in \mathbb{S}_2$ can be written as

$$g = a_0 + a_1S + a_2S^2 + \cdots$$

with

$$a_i \in \{0, 1, \omega, \omega^2\}$$

and $a_0 \neq 0$. Let

$$S_2 = \left\{ \sum_{i\geq 0} a_iS^i \in \mathbb{S}_2 \ :\ a_0 = 1 \right\}$$

denote the 2-Sylow subgroup of $\mathbb{S}_2$. The version of the Morava stabilizer algebra $(\mathbb{F}_4[u^{\pm 1}], \Sigma_2)$ introduced at the beginning of this section can be regarded as an algebra of functions on S_2:

$$\Sigma_2 \cong \mathrm{Map}^c(S_2, (K_2)_*) \cong \mathbb{F}_4[u^{\pm 1}][t_1, t_2, \ldots]/(t_k^4 - v_2^{2^k-1} t_k).$$

Here, the functions t_k are defined as

$$t_k(1 + a_1 S + a_2 S^2 + \ldots) = a_k u^{1-2^k}.$$

The coproduct ψ is determined by $\psi(t_k) = \sum t'_k \otimes t''_k$ where

$$t_k(g'g'') = \sum t'_k(g') t''_k(g''), \qquad g', g'' \in S_2.$$

The inclusion of G_{24} in $\mathbb{S}_2$ gives a splitting of the short exact sequence

$$1 \to K \to \mathbb{S}_2 \to G_{24} \to 1$$

where K is the open normal subgroup of $\mathbb{S}_2$

$$K = \{1 + a_2 S^2 + a_3 S^3 + \cdots \in \mathbb{S}_2 \ : \ a_2 \in \{0, \omega\}\}$$

discussed at length in Section 2.5 of [Bea15].

The inclusion of groups

$$K \hookrightarrow S_2$$

corresponds to a quotient of Hopf algebras

$$\Sigma_2 \to \overline{\Sigma}_2$$

where

$$\overline{\Sigma}_2 = \Sigma_2/(t_1, \omega v_2 t_2 + t_2^2)$$

(compare with [Rav86, Proposition 6.3.30], but his choice of K is Galois conjugate to ours).

4.2. **The Morava module of Z.** In this subsection we will use the computations of [BE16b] to derive the following result (where G_{48} is the group (4.1.3)).

Proposition 4.2.1. *There is an isomorphism of G_{48}-modules*

$$(E_2)_* Z \cong \mathrm{CoInd}_{C_3 \rtimes \mathrm{Gal}}^{G_{48}} \mathbb{F}_4[u^{\pm 1}]$$

where $C_3 \rtimes \mathrm{Gal}$ acts on $\mathbb{F}_4[u^{\pm 1}]$ via

$$\omega_*(\lambda u^k) = \lambda \omega^k u^k, \qquad \sigma_*(\lambda u^k) = \lambda^\sigma u^k.$$

The proof of this proposition will require some preliminary recollections from [BE16b]. Let E'_2 denote the Morava E-theory spectrum associated to the Honda height 2 formal group over $\mathbb{F}_4$. The spectrum E'_2 has an action of the extended Morava stabilizer group $\mathbb{G}'_2 = \mathbb{S}_2 \rtimes \mathrm{Gal}'$ of the previous subsection.

The third author and Egger computed $(E'_2)_* Z$ as

$$(E'_2)_* Z \cong \mathbb{F}_4[u^{\pm}]\{\bar{x}_0, \bar{x}_2, \bar{x}_4, \bar{x}_6, \bar{y}_6, \bar{y}_8, \bar{y}_{10}, \bar{y}_{12}\}, \quad |\bar{x}_i| = |\bar{y}_i| = 0, \tag{4.2.2}$$

with an explicit action of $\mathbb{S}_2$ [BE16b, Table 1]. Since the generators $u^{i/2}\bar{x}_i$ and $u^{i/2} y_i$ are in the image of the map

$$BP_* Z \to (E'_2)_* Z,$$

they have trivial action of the Galois group Gal', and therefore Gal' acts on (4.2.2) by acting on $\mathbb{F}_4$. Following the proof of [BE16b, Thm. 4.12], we see that for any $x \in (E_2')_0 Z$ with

$$x = \bar{y}_{12} + \alpha_0 \bar{x}_0 + \alpha_2 \bar{x}_2 + \alpha_4 \bar{x}_4 + \alpha_6 \bar{x}_6, \quad \alpha_i \in \mathbb{F}_4 \tag{4.2.3}$$

we have[4]

$$(E_2')_0 Z = \mathbb{F}_4[Q_8]\{x\}.$$

Proof of Proposition 4.2.1. Let $\bar{E}_2$ denote the Morava E-theory associated to the height 2 Honda formal group over the algebraic closure $\bar{\mathbb{F}}_2$, with action of

$$\bar{\mathbb{G}}_2' = \mathbb{S}_2 \rtimes \mathrm{Gal}(\bar{\mathbb{F}}_2/\mathbb{F}_2).$$

Let σ' denote the Frobenius, regarded as a generator of $\mathrm{Gal}(\bar{\mathbb{F}}_2/\mathbb{F}_2)$, acting on $\mathbb{S}_2$ as in the previous subsection. Then we have

$$E_2' \simeq \bar{E}_2^{h\langle(\sigma')^2\rangle}.$$

Since the formal group of the elliptic curve C is isomorphic to the Honda formal group over $\bar{\mathbb{F}}_2$, we deduce that the associated Morava E-theory is the same, but the action of the Galois group is different. The calculations of the previous subsection imply that if we define

$$\sigma := \alpha\sigma' \in \bar{\mathbb{G}}_2'$$

then the Morava E-theory associated to the formal group of C over $\mathbb{F}_4$ is given by

$$E_2 \simeq \bar{E}_2^{h\langle(\sigma)^2\rangle}.$$

Since $\sigma^4 = (\sigma')^4$, we deduce that E_2 and E_2' have the common extension

$$E_2'' := \bar{E}_2^{h\langle\sigma^4\rangle}.$$

We therefore have

$$(E_2'')_0 Z = \mathbb{F}_{16} \otimes_{\mathbb{F}_4} (E_2')_0 Z \cong \mathbb{F}_{16}[Q_8]\{x\}$$

for any x of the form (4.2.3) (with $\alpha_i \in \mathbb{F}_{16}$). Let $\widetilde{\omega} \in \mathbb{F}_{16}^\times$ be a generator, so that

$$\widetilde{\omega}^{\sigma^4} = \widetilde{\omega}^{16} = \widetilde{\omega}.$$

Since $\widetilde{\omega} + \widetilde{\omega}^4 \in \mathbb{F}_4$ we can take $\widetilde{\omega}$ so that

$$\widetilde{\omega} + \widetilde{\omega}^4 = \omega \in \mathbb{F}_4.$$

Define

$$x := \bar{y}_{12} + (1 + \widetilde{\omega}^4 + \widetilde{\omega}^8)\bar{x}_6 + (a+b)(\widetilde{\omega} + \widetilde{\omega}^8)\bar{x}_0$$

(where $a, b \in \mathbb{F}_2$ are those associated to the choice of $Z \in \widetilde{\mathcal{Z}}$ as in [BE16b, Lem. 3.5]). Then it follows from [BE16b, Table 1] and

$$\alpha = 1 + 2\omega \mod 4$$

that

(1) $\sigma = \alpha\sigma'$ acts trivially on x,
(2) $\langle\omega\rangle = C_3 < \mathbb{S}_2$ acts trivially on x,
(3) x generates $(E_2'')_0 Z$ as a free $\mathbb{F}_{16}[Q_8]$-module.

[4] In the notation of [BE16b], we have $x = k \cdot c_3' + \text{terms involving } c_i$, where $k \in Q_8$ is the unit quaternion.

It follows that x generates

$$(E_2)_0 Z \cong [(E_2'')_0 Z]^{\sigma^2=1}$$

as an $\mathbb{F}_4[Q_8]$-module. This, together with (1) and (2) above, implies the desired result. □

4.3. **The good complex as a subcomplex of the cobar complex of $\overline{\Sigma}_2$.** The map tmf → TMF induces a map of spectral sequences

$$ {}^{\mathrm{tmf}}E_r^{*,*}(Z) \to {}^{\mathrm{TMF}}E_r^{*,*}(Z). \tag{4.3.1}$$

The kernel of ${}^{\mathrm{tmf}}E_1^{*,*}(Z) \to {}^{\mathrm{TMF}}E_1^{*,*}(Z)$ is $V^{*,*}(Z)$ and the image is

$$\mathcal{C}^{*,*}(Z) \subseteq {}^{\mathrm{TMF}}E_1^{*,*}(Z).$$

Let E_2 be the Morava E-theory spectrum associated to the formal group $\widehat{C}$ over $\mathbb{F}_4$. Then the Goerss-Hopkins-Miller theorem implies that E_2 has an action of $\mathbb{G}_2$, and we have

$$L_{K(2)}\mathrm{TMF} \simeq E_2^{hG_{48}}$$

where G_{48} is the group defined in (4.1.3). We will now explain how the complex ${}^{\mathrm{TMF}}E_1(Z)$ can be regarded as a subcomplex of the cobar complex for the Hopf algebra $\overline{\Sigma}_2$.

The first step will be to express the E_1-term in terms of the Morava stabilizer group (Corollary 4.3.4).

Lemma 4.3.2. *There is a $\mathbb{G}_2$-equivariant isomorphism*

$$(E_2)_*(\mathrm{TMF} \wedge Z) \cong \mathrm{Map}^c(\mathbb{G}_2/G_{48}, (E_2)_*Z)$$

(where $\mathbb{G}_2$ acts on Map^c by the conjugation action on functions), and this leads to an isomorphism

$$\pi_*\mathrm{TMF} \wedge \mathrm{TMF} \wedge Z \cong \mathrm{Map}^c_{C_3 \rtimes \mathrm{Gal}}(\mathbb{G}_2/G_{48}, \mathbb{F}_4[u^{\pm 1}])$$

where $\mathrm{Map}^c_{C_3 \rtimes \mathrm{Gal}}(\mathbb{G}_2/G_{48}, \mathbb{F}_4[u^{\pm 1}])$ denotes the $C_3 \rtimes \mathrm{Gal}$ equivariant maps.

Proof. Since Z is a type 2 complex, $X \wedge Z$ is $K(2)$-local for any $E(2)$-local spectrum X. In particular, we have

$$\mathrm{TMF} \wedge Z \simeq E_2^{hG_{48}} \wedge Z. \tag{4.3.3}$$

Using the fact that for finite groups, homotopy fixed points and homotopy orbits of $K(2)$-local spectra are $K(2)$-locally equivalent [Kuh04] we get

$$\mathrm{TMF} \wedge \mathrm{TMF} \wedge Z \simeq E_2^{hG_{48}} \wedge E_2^{hG_{48}} \wedge Z \simeq (E_2 \wedge (E_2^{hG_{48}} \wedge Z))^{hG_{48}}.$$

We use the homotopy fixed point spectral sequence

$$H^s(G_{48}, (E_2)_t(E_2^{hG_{48}} \wedge Z)) \Longrightarrow \pi_{t-s}\mathrm{TMF} \wedge \mathrm{TMF} \wedge Z.$$

By [BBGS18, Corollary 2.1],

$$(E_2)_*(E_2^{hG_{48}} \wedge Z) \cong (E_2)_*(E_2 \wedge Z)^{hG_{48}} \cong \mathrm{Map}^c(\mathbb{G}_2/G_{48}, (E_2)_*Z)$$

with action of G_{48} given by the conjugation action on functions. Since we have an isomorphism of G_{48}-modules

$$(E_2)_*Z \cong \mathrm{CoInd}_{C_3 \rtimes \mathrm{Gal}}^{G_{48}} \mathbb{F}_4[u^{\pm 1}] \cong \mathrm{Map}_{C_3 \rtimes \mathrm{Gal}}(G_{48}, \mathbb{F}_4[u^{\pm 1}])$$

it follows that

$$(E_2)_*(E_2 \wedge Z)^{hG_{48}} \cong \mathrm{Map}_{C_3 \rtimes \mathrm{Gal}}(G_{48}, \mathrm{Map}^c(\mathbb{G}_2/G_{48}, \mathbb{F}_4[u^{\pm 1}])).$$

In particular, the E_2-term of the homotopy fixed point spectral sequence is

$$H^*(G_{48}, (E_2)_*(E_2 \wedge Z)^{hG_{48}}) \cong H^*(C_3 \rtimes \mathrm{Gal}, \mathrm{Map}^c(\mathbb{G}_2/G_{48}, \mathbb{F}_4[u^{\pm 1}])).$$

Since C_3 has order coprime to 2 and Gal acts freely on $\mathbb{F}_4$, the E_2-term is concentrated in degree $s = 0$, and given by

$$\mathrm{Map}^c(\mathbb{G}_2/G_{48}, \mathbb{F}_4[u^{\pm 1}])^{C_3 \rtimes \mathrm{Gal}}.$$

The spectral sequence collapses, giving the result. □

Corollary 4.3.4. *For $s \geq 1$, there is a $\mathbb{G}_2$-equivariant isomorphism*

$$(E_2)_*(\mathrm{TMF}^{\wedge s} \wedge Z) \cong \mathrm{Map}^c((\mathbb{G}_2/G_{48})^{\times s}, (E_2)_*Z)$$

with the diagonal action on $(\mathbb{G}_2/G_{48})^{\times s}$ and action on Map^c the conjugation action on functions. This leads to an isomorphism

$$^{\mathrm{TMF}}E_1^{s,*}(Z) \cong \pi_*\mathrm{TMF}^{\wedge s+1} \wedge Z \cong \mathrm{Map}^c_{C_3 \rtimes \mathrm{Gal}}(\underbrace{\mathbb{G}_2 \times_{G_{48}} \cdots \times_{G_{48}} \mathbb{G}_2}_{s}/G_{48}, \mathbb{F}_4[u^{\pm 1}]).$$

The action on

$$\mathbb{G}_2 \times_{G_{48}} \cdots \times_{G_{48}} \mathbb{G}_2/G_{48}$$

is via by left multiplication on the first factor of $\mathbb{G}_2$.

Proof. Suppose that the claim holds for $s-1$. Then

$$\begin{aligned} E_2 \wedge \mathrm{TMF}^{\wedge s} \wedge Z &\simeq E_2 \wedge E_2^{hG_{48}} \wedge \mathrm{TMF}^{\wedge(s-1)} \wedge Z \\ &\simeq (E_2 \wedge E_2 \wedge \mathrm{TMF}^{\wedge(s-1)} \wedge Z)^{hG_{48}} \end{aligned}$$

where G_{48} acts on the second copy of E_2. The E_2-page of the homotopy fixed point spectral sequence is given by

$$H^*(G_{48}, (E_2)_*(E_2 \wedge \mathrm{TMF}^{\wedge(s-1)} \wedge Z)).$$

Furthermore,

$$\begin{aligned} (E_2)_*(E_2 \wedge \mathrm{TMF}^{\wedge(s-1)} \wedge Z) &\cong \mathrm{Map}^c(\mathbb{G}_2, (E_2)_*\mathrm{TMF}^{\wedge(s-1)} \wedge Z) \\ &\cong \mathrm{Map}^c(\mathbb{G}_2, \mathrm{Map}^c((\mathbb{G}_2/G_{48})^{\times(s-1)}, (E_2)_*Z)). \end{aligned}$$

It follows that

$$\begin{aligned} H^*(G_{48}, (E_2)_*(E_2 \wedge \mathrm{TMF}^{\wedge(s-1)} \wedge Z)) &\cong H^0(G_{48}, (E_2)_*(E_2 \wedge \mathrm{TMF}^{\wedge(s-1)} \wedge Z)) \\ &\cong \mathrm{Map}^c((\mathbb{G}_2/G_{48})^{\times s}, (E_2)_*Z). \end{aligned}$$

which proves the first claim.

Next,

$$\mathrm{TMF}^{\wedge(s+1)} \wedge Z \simeq (E_2 \wedge \mathrm{TMF}^{\wedge s} \wedge Z)^{hG_{48}}.$$

We use the homotopy fixed point spectral sequence again, together with the fact that

$$\begin{aligned}(E_2)_*(\mathrm{TMF}^{\wedge s} \wedge Z) &\cong \mathrm{Map}^c((\mathbb{G}_2/G_{48})^{\times s}, (E_2)_*Z) \\ &\cong \mathrm{Map}^c((\mathbb{G}_2/G_{48})^{\times s}, \mathrm{Map}_{C_3 \rtimes \mathrm{Gal}}(G_{48}, \mathbb{F}_4[u^{\pm 1}])) \\ &\cong \mathrm{Map}_{C_3 \rtimes \mathrm{Gal}}(G_{48}, \mathrm{Map}^c((\mathbb{G}_2/G_{48})^{\times s}, \mathbb{F}_4[u^{\pm 1}])).\end{aligned}$$

The proof is finished in a way analogous to that of Lemma 4.3.2. The last step identifies

$$\mathrm{Map}^c_{C_3 \rtimes \mathrm{Gal}}((\mathbb{G}_2/G_{48})^{\times s}, \mathbb{F}_4[u^{\pm 1}]) \cong \mathrm{Map}^c_{C_3 \rtimes \mathrm{Gal}}(\underbrace{\mathbb{G}_2 \times_{G_{48}} \cdots \times_{G_{48}} \mathbb{G}_2}_{s}/G_{48}, \mathbb{F}_4[u^{\pm 1}])$$

via a shearing isomorphism. □

It is not clear how the groups

$$\mathrm{Map}^c_{C_3 \rtimes \mathrm{Gal}}(\mathbb{G}_2^{\times_{G_{48}} s}/G_{48}, \mathbb{F}_4[u^{\pm 1}])$$

in Corollary 4.3.4 form a cochain complex. We now will address this by showing that they are a subcomplex of the E_2-based Adams spectral sequence for Z.

The map of spectra $\mathrm{TMF} \to E_2$ induces a map of Adams spectral sequences. The induced map on E_1-terms

$${}^{\mathrm{TMF}}E_1(Z) \to {}^{E_2}E_1(Z)$$

is given by the canonical inclusion

$$\begin{aligned}\mathrm{Map}^c_{C_3 \rtimes \mathrm{Gal}}(\mathbb{G}_2^{\times_{G_{48}} s}/G_{48}, \mathbb{F}_4[u^{\pm 1}]) &\cong \mathrm{Map}^c_{G_{48}}(\mathbb{G}_2^{\times_{G_{48}} s}/G_{48}, \mathrm{CoInd}_{C_3 \rtimes \mathrm{Gal}}^{G_{48}} \mathbb{F}_4[u^{\pm 1}]) \\ &\subseteq \mathrm{Map}^c(\mathbb{G}_2^s, \mathrm{CoInd}_{C_3 \rtimes \mathrm{Gal}}^{G_{48}} \mathbb{F}_4[u^{\pm 1}])\end{aligned}$$

where the latter is the cobar complex for $\mathbb{G}_2$ acting on $(E_2)_*Z$:

$$C^s_{\mathbb{G}_2}((E_2)_*Z) \cong {}^{E_2}E_1^{s,*}(Z).$$

In particular, the differential in the cobar complex for $\mathbb{G}_2$ restricts to give the differential on the subcomplex

$$\mathrm{Map}^c_{C_3 \rtimes \mathrm{Gal}}(\mathbb{G}_2^{\times_{G_{48}} s}/G_{48}, \mathbb{F}_4[u^{\pm 1}]) \subseteq \mathrm{Map}^c(\mathbb{G}_2^s, \mathrm{CoInd}_{C_3 \rtimes \mathrm{Gal}}^{G_{48}} \mathbb{F}_4[u^{\pm 1}]).$$

We now have the following lemma.

Lemma 4.3.5. *There is an embedding of cochain complexes*

$${}^{\mathrm{TMF}}E_1(Z) \subset C^*_{\overline{\Sigma}_2}((K_2)_*).$$

Proof. The injection comes from the composite

$$\begin{aligned}{}^{\mathrm{TMF}}E_1^{s,*}(Z) &\cong \mathrm{Map}^c_{C_3 \rtimes \mathrm{Gal}}(\mathbb{G}_2^{\times_{G_{48}} s}/G_{48}, \mathbb{F}_4[u^{\pm 1}]) \\ &\hookrightarrow \mathrm{Map}^c(\mathbb{G}_2^{\times_{G_{48}} s}/G_{48}, \mathbb{F}_4[u^{\pm 1}]) \\ &\cong \mathrm{Map}^c(K^s, \mathbb{F}_4[u^{\pm 1}]) \\ &\cong C^s_{\overline{\Sigma}_2}((K_2)_*).\end{aligned}$$

Here, the second to last isomorphism comes from the fact that the composite

$$K \to \mathbb{G}_2 \to \mathbb{G}_2/G_{48}$$

is a homeomorphism. $\square$

4.4. **The sub-Hopf algebra $\widetilde{\Sigma}(2) \subset \overline{\Sigma}_2$.** We shall now study a sub-Hopf algebra $(K(2)_*, \widetilde{\Sigma}(2))$ of the Hopf algebra $((K_2)_*, \overline{\Sigma}_2)$ such that the image of ${}^{\mathrm{TMF}}E_1(Z)$ in the cobar complex for $\overline{\Sigma}_2$ is the cobar complex for $\widetilde{\Sigma}(2)$.

Define Hopf algebras

$$\widetilde{\Sigma}(2) \subset \overline{\Sigma}(2) \subset \overline{\Sigma}_2$$

by letting $\widetilde{\Sigma}(2)$ be the image of the map

$$\mathrm{Map}^c_{C_3 \rtimes \mathrm{Gal}}(\mathbb{G}_2/G_{48}, \mathbb{F}_4[u^{\pm 1}]) \hookrightarrow \mathrm{Map}^c(K, \mathbb{F}_4[u^{\pm 1}]) \cong \overline{\Sigma}_2.$$

and letting $\overline{\Sigma}(2)$ be the image of the map

$$\mathrm{Map}^c_{C_3}(\mathbb{G}_2/G_{48}, \mathbb{F}_4[u^{\pm 1}]) \hookrightarrow \mathrm{Map}^c(K, \mathbb{F}_4[u^{\pm 1}]) \cong \overline{\Sigma}_2.$$

Note that there is a nonstandard induced $C_3 \rtimes \mathrm{Gal}$ action on $\overline{\Sigma}_2$ so that

$$\begin{aligned} \overline{\Sigma}(2) &= \overline{\Sigma}_2^{C_3}, \\ \widetilde{\Sigma}(2) &= \overline{\Sigma}(2)^{\mathrm{Gal}} = \overline{\Sigma}_2^{C_3 \rtimes \mathrm{Gal}}. \end{aligned}$$

We now compute this action of $C_3 \rtimes \mathrm{Gal}$ on

$$\overline{\Sigma}_2 = \mathbb{F}_4[u^{\pm 1}][\bar{t}_2, \bar{t}_3, \cdots]/(\bar{t}_2^2 + \omega v_2 \bar{t}_2, \bar{t}_k^4 + v_2^{2^k - 1}\bar{t}_k).$$

Here we use $\bar{t}_k$ to denote the image of $t_k \in \Sigma_2$ in $\overline{\Sigma}_2$. Let σ be the generator of Gal, and we will denote the generator of $C_3 \subset \mathbb{G}_2$ by ω, our fixed choice of 3rd root of unity.

Recall [Bea15] that elements $x \in K$ can be written as

$$x = 1 + a_2 S^2 + a_3 S^3 + \cdots$$

with $a_2 \in \{0, \omega\}$ and $a_i \in \{0, 1, \omega, \omega^2\}$ for $i > 2$. The function

$$\bar{t}_i \in \overline{\Sigma}_2 = \mathrm{Map}^c(K, \mathbb{F}_4[u^{\pm 1}])$$

is given on elements x as above by the formula

$$\bar{t}_i(x) = a_i u^{1-2^i}.$$

Under the isomorphism

$$\mathrm{Map}^c(K, \mathbb{F}_4[u^{\pm 1}]) \cong \mathrm{Map}^c(\mathbb{G}_2/G_{48}, \mathbb{F}_4[u^{\pm 1}])$$

the function $\bar{t}_i$ is given on a coset gG_{48} by

$$\bar{t}_i(gG_{48}) = t_i(x)$$

where x is the unique element of K so that $xG_{48} = gG_{48}$.

Note that C_3 acts on $\mathbb{F}_4[u^{\pm 1}]$ through ring maps by the formula

$$\omega \cdot u = \omega u$$

and Gal acts through the Galois action on $\mathbb{F}_4$, so

$$\mathbb{F}_4[u^{\pm 1}]^{C_3 \rtimes \mathrm{Gal}} = \mathbb{F}_2[v_2^{\pm 1}].$$

Lemma 4.4.1. *The functions $\bar{t}_k \in \overline{\Sigma}_2$ are C_3 equivariant, so the C_3 action on $\bar{t}_k$ is trivial.*

Proof. We have (for $a_2 \in \{0, \omega\}$):

$$\begin{aligned}
\bar{t}_k(\omega(1 + a_2 S^2 + a_3 S^3 + \cdots)G_{48}) &= \bar{t}_k((\omega + \omega a_2 S^2 + \omega a_3 S^3 + \cdots)G_{48}) \\
&= \bar{t}_k((\omega + \omega a_2 S^2 + \omega a_3 S^3 + \cdots)\omega^2 G_{48}) \\
&= \bar{t}_k((1 + a_2 S^2 + \omega^2 a_3 S^3 + \cdots)G_{48}) \\
&= \begin{cases} a_k u^{1-2^k}, & k \text{ even}, \\ \omega^2 a_k u^{1-2^k}, & k \text{ odd} \end{cases} \\
&= \omega \cdot a_k u^{1-2^k} \\
&= \omega \cdot \bar{t}_k((1 + a_2 S^2 + a_3 S^3 + \cdots)G_{48}). \qquad \square
\end{aligned}$$

Corollary 4.4.2. *The sub-Hopf algebra $\overline{\Sigma}(2) \subset \overline{\Sigma}_2$ is given by*

$$\overline{\Sigma}(2) = \mathbb{F}_4[v_2^{\pm 1}][\bar{t}_2, \bar{t}_3, \cdots]/(\bar{t}_2^2 + \omega v_2 \bar{t}_2, \bar{t}_k^4 + v_2^{2^k - 1}\bar{t}_k).$$

Lemma 4.4.3. *We have*

$$\sigma \cdot \bar{t}_2 = \omega \bar{t}_2$$

and the element $\widetilde{t}_2 := \omega^2 \bar{t}_2 \in \overline{\Sigma}(2)$ is Galois invariant.

Proof. We compute the conjugation action (for $a_2 \in \{0, \omega\}$) using that $\sigma^{-1} = \sigma$, Lemma 4.1.5, and the fact that $\alpha \equiv 1 \pmod 2$:

$$\begin{aligned}
(\sigma \cdot \bar{t}_2)((1 + a_2 S^2 + \cdots)G_{48}) &= \sigma[\bar{t}_2(\sigma(1 + a_2 S^2 + \cdots)G_{48})] \\
&= \sigma[\bar{t}_2(-\alpha(1 + a_2^\sigma S^2 + \cdots)\alpha^\sigma G_{48})] \\
&= \sigma[\bar{t}_2((1 + a_2^\sigma S^2 + \cdots)G_{48})] \\
&= \begin{cases} \sigma[\bar{t}_2((1 + 0S^2 + \cdots)G_{48})], & a_2 = 0, \\ \sigma[\bar{t}_2((1 + \omega^2 S^2 + \cdots)G_{48})], & a_2 = \omega. \end{cases}
\end{aligned}$$

Now if $a_2 = 0$, it follows we have

$$\begin{aligned}
\sigma[\bar{t}_2(\sigma(1 + 0S^2 + \cdots)G_{48})] &= 0 \\
&= \omega \bar{t}_2((1 + 0S^2 + \cdots)G_{48}).
\end{aligned}$$

However, if $a_2 = \omega$, the coset representative is not in K, and we have to rectify this by adjusting it by right multiplication with

$$-1 = 1 + S^2 + S^4 + \cdots \in \mathbb{G}_2$$

to get it into K. We have

$$\begin{aligned}
\sigma[\bar{t}_2((1 + \omega^2 S^2 + \cdots)G_{48})] &= \sigma[\bar{t}_2((1 + \omega^2 S^2 + \cdots)(-1)G_{48})] \\
&= \sigma[t_2((1 + \omega^2 S^2 + \cdots)(-1))] \\
&= \sigma[\omega u^{-3}] \\
&= \omega^2 u^{-3} \\
&= \omega \bar{t}_2((1 + \omega S^2 + \cdots)G_{48}). \qquad \square
\end{aligned}$$

Define $\widetilde{\sigma}(2)$ to be the image of the composite

$$\pi_* \mathrm{tmf} \wedge \mathrm{tmf} \wedge Z \to \pi_* \mathrm{TMF} \wedge \mathrm{TMF} \wedge Z \hookrightarrow \overline{\Sigma}(2).$$

Lemma 4.4.4. *The Hopf algebra structure on* $(\mathbb{F}_4[v_2^{\pm 1}], \overline{\Sigma}(2))$ *restricts to a Hopf algebra structure on* $(k(2)_*, \widetilde{\sigma}(2))$.

Proof. The only thing which is not obvious is that the coproduct of $\overline{\Sigma}(2)$ restricts to a coproduct on $\widetilde{\sigma}(2)$. Using the fact that $\mathrm{tmf} \wedge Z \simeq k(2)$, it suffices to consider the diagram, where ℓ is the unit:

$$\begin{tikzcd}
k(2)_*(S \wedge \mathrm{tmf}) \arrow[r] \arrow[d, "(\ell\wedge 1)_*"'] & K(2)_*(S \wedge \mathrm{TMF}) \arrow[r] \arrow[d, "(\ell\wedge 1)_*"'] & \overline{\Sigma}(2) \arrow[dd, "\psi"] \\
k(2)_*(\mathrm{tmf} \wedge \mathrm{tmf}) \arrow[r, "(1)"] & K(2)_*(\mathrm{TMF} \wedge \mathrm{TMF}) & \\
k(2)_*(\mathrm{tmf}) \otimes_{k(2)_*} k(2)_*\mathrm{tmf} \arrow[u, "(*)"] \arrow[r, "(2)"'] & K(2)_*(\mathrm{TMF}) \otimes_{K(2)_*} K(2)_*\mathrm{TMF} \arrow[u, "\cong"'] \arrow[r] & \overline{\Sigma}(2) \otimes_{\mathbb{F}_4[v_2^{\pm 1}]} \overline{\Sigma}(2)
\end{tikzcd}$$

Since $(*)$ is an isomorphism after inverting v_2, it follows that maps (1) and (2) have isomorphic images. The result follows. $\square$

Theorem 4.4.5. *The Hopf algebra* $\widetilde{\sigma}(2) \subset \overline{\Sigma}(2)$ *has the form*

$$\widetilde{\sigma}(2) = \mathbb{F}_2[v_2^{\pm 1}][\widetilde{t_2^2}, \widetilde{t}_3, \ldots]/((\widetilde{t_2^2})^2 = v_2^2 \widetilde{t_2^2},\ \widetilde{t}_k^4 = \textit{terms with Adams filtration} > 0)$$

where $\widetilde{t_2^2} = (\omega^2 \bar{t}_2)^2$ *and for* $k \geq 3$

$$\widetilde{t}_k = \bar{t}_k + \textit{terms of higher Adams filtration.}$$

There is an isomorphism of cochain complexes

$$\mathcal{C}^{*,*}(Z) \cong C^*_{\widetilde{\sigma}(2)}(k(2)_*).$$

Proof. By Lemma 4.3.5, it suffices to establish that the image of the map

$$\pi_* \mathrm{tmf}^{\wedge n+1} \wedge Z \to {}^{\mathrm{TMF}}E_1(Z) \hookrightarrow C^*_{\overline{\Sigma}(2)}(K(2)_*)$$

is what we claim it is. We focus on the case of $n = 2$; it will be apparent that the general case is essentially the same. Recall that we have

$$\begin{aligned}
{}^{ass}E_2(\mathrm{tmf} \wedge \mathrm{tmf} \wedge Z) &\cong {}^{ass}E_2(k(2) \wedge \mathrm{tmf}) \\
&\cong \mathbb{F}_2[v_2][\zeta_2^4, \zeta_3^2, \zeta_4^2, \ldots]/(\zeta_i^8) \\
&\quad \oplus \text{simple } v_2\text{-torsion in Adams filtration } 0.
\end{aligned}$$

Note that since the elements ζ_2^4, ζ_i^2 all lie in Adams filtration zero, *the Adams filtrations and v_2-Bockstein filtrations on $k(2)_*\mathrm{tmf}$ agree.* This means that an element in $K(2)_*\mathrm{tmf} \cong K(2)_*\mathrm{TMF}$ is in the image of the map

$${}^{\mathrm{tmf}}E_1^{1,*}(Z) \cong k(2)_*\mathrm{tmf} \to v_2^{-1}k(2)_*\mathrm{tmf} \cong K(2)_*\mathrm{TMF} \cong {}^{\mathrm{TMF}}E_1^{1,*}(Z)$$

if and only if it is detected (in the localized Adams spectral sequence) by an element in the image of the map

$${}^{ass}E_2(k(2)_*\mathrm{tmf}) \to v_2^{-1}\, {}^{ass}E_2(k(2)_*\mathrm{tmf}).$$

For the purposes of this paper, define [LN12], [AL17]

$$BP\langle 2\rangle := \mathrm{tmf}_1(3)$$

where we have

$$\mathrm{tmf}_1(3)_{K(2)} \simeq E_2^{hC_3 \rtimes \mathrm{Gal}}.$$

Consider the commutative diagram

$$\begin{tikzcd} k(2)_*\mathrm{tmf} \arrow[r] \arrow[d, "(1)"'] & K(2)_*\mathrm{TMF} \arrow[r, hook] \arrow[d] & \mathrm{Map}^c(\mathbb{G}_2/G_{48}, \mathbb{F}_4[u^{\pm 1}]) \arrow[d, hook, "(3)"] \\ k(2)_*BP\langle 2\rangle \arrow[r, "(2)"'] & K(2)_*E_2 \arrow[r, hook] & \mathrm{Map}^c(\mathbb{G}_2, \mathbb{F}_4[u^{\pm 1}]) \end{tikzcd} \tag{4.4.6}$$

We wish to determine which v_2 multiple of $\widetilde{t}_2$ is in positive Adams filtration. To that end, we must compute the image of $\widetilde{t}_2$ under map (3) in (4.4.6). This is tantamount to computing, for $g \in \mathbb{G}_2$, the value $\widetilde{t}_2(gG_{48})$. Since we have already established $\widetilde{t}_2$ is $C_3 \rtimes$ Gal-equivariant, we may assume

$$g = 1 + a_1 S + a_2 S^2 + \cdots .$$

Write $a_1 = \alpha\omega + \beta\omega^2$ with $\alpha, \beta \in \mathbb{F}_2$. Using the fact that the elements j and k in G_{48} are given by

$$\begin{aligned} j &= 1 + \omega^2 S + \omega S^2 + \cdots \\ k &= 1 + \omega S + \omega S^2 + \cdots \end{aligned}$$

(see [Bea15]) we compute:

$$\begin{aligned} \widetilde{t}_2(gG_{48}) &= \widetilde{t}_2((1 + (\alpha\omega + \beta\omega^2)S + a_2 S^2 + \cdots)G_{48}) \\ &= \widetilde{t}_2((1 + (\alpha\omega + \beta\omega^2)S + a_2 S^2 + \cdots)k^\alpha j^\beta G_{48}) \\ &= \widetilde{t}_2((1 + (a_2 + (\alpha+\beta)\omega^2 + \alpha\beta\omega)S^2 + \cdots)G_{48}). \end{aligned}$$

Let

$$\mathrm{Tr}, \mathrm{N} : \mathbb{F}_4 \to \mathbb{F}_2$$

be the trace and norm, respectively, so that $\mathrm{Tr}(a) = a + a^\sigma$ and $\mathrm{N}(a) = aa^\sigma$. From the definition of $\widetilde{t}_2$ we find

$$\widetilde{t}_2((1 + a_2 S^2 + \cdots)G_{48}) = \mathrm{Tr}(a_2)u^{-3}.$$

It follows from the above calculation that

$$\widetilde{t}_2((1 + a_1 S + a_2 S^2 + \cdots)G_{48}) = (\mathrm{Tr}(a_2) + \mathrm{N}(a_1))u^{-3}.$$

Thus the image of $\widetilde{t}_2$ under map (3) in (4.4.6) is the image of

$$t_2 + t_2^2 v_2^{-1} + t_1^3$$

under map (2). Since the elements $t_i \in k(2)_*BP\langle 2\rangle$ all have Adams filtration 0, it follows that $v_2\widetilde{t}_2 = \widetilde{t}_2^2 \in K(2)_*\mathrm{TMF}$ lifts to an element

$$\widetilde{t}_2^2 = v_2 t_2 + t_2^2 + v_2 t_1^3 \tag{4.4.7}$$

of $k(2)_*\mathrm{tmf}$.

For $k \geq 3$, we *define* $\widetilde{t}_k \in \widetilde{\sigma}(2)$ to be the image of an element of $k(2)_*\mathrm{tmf}$ detected by ζ_k^2. Since in the Adams spectral sequence for $k(2)_*BP\langle 2\rangle$ the element ζ_k^2 detects t_k, we deduce that the image of $\widetilde{t}_k$ under (1) satisfies

$$\widetilde{t}_k = t_k + \text{terms of positive Adams filtration.}$$

The result for $n = 2$ follows.

Similar reasoning shows that the image of

$$^{\mathrm{tmf}}E_1^{n,*}(Z) \cong k(2)_*\mathrm{tmf}^{\wedge n} \to K(2)_*\mathrm{TMF}^{\wedge n} = \widetilde{\Sigma}(2)^{\otimes_{K(2)_*} n} \cong {}^{\mathrm{TMF}}E_1^{n,*}(Z)$$

is $\widetilde{\sigma}(2)^{\otimes_{k(2)_*} n}$. □

Note that while we do not know the full structure of $\widetilde{\sigma}(2)$ because of the complicated action of Gal on $\overline{\Sigma}(2)$, we do completely know the structure of $\overline{\sigma}(2) := \widetilde{\sigma}(2) \otimes \mathbb{F}_4 \subset \overline{\Sigma}(2)$.

$$\overline{\sigma}(2) = \mathbb{F}_4[v_2][\widetilde{t_2^2}, \bar{t}_3, \cdots]/((\widetilde{t_2^2})^2 + v_2^2\widetilde{t_2^2}, \bar{t}_k^4 + v_2^{2^k-1}\bar{t}_k).$$

5. The cohomology of the good complex

In the previous section we established that

$$\mathcal{C}^{*,*}(Z) \cong C^*_{\widetilde{\sigma}(2)}(k(2)_*).$$

In this section we will compute the E_1-term of a spectral sequence which computes the cohomology

$$H^*(\widetilde{\sigma}(2)) := H(C^*_{\widetilde{\sigma}(2)}(k(2)_*)) \cong H(\mathcal{C}^{*,*}(Z)) =: H^{*,*}(\mathcal{C}).$$

In our low dimensional range, it will turn out that there are no possible differentials in this spectral sequence.

5.1. **Overview of the strategy.**

Recall from the previous section that we really only have a complete understanding of the base change

$$\overline{\sigma}(2) := \widetilde{\sigma}(2) \otimes \mathbb{F}_4$$

and we only know the generators of $\widetilde{\sigma}(2)$ in $\overline{\sigma}(2)$ modulo terms of higher Adams filtration. Our approach to understanding this cohomology will be to understand aspects of the cohomology of $\overline{\sigma}(2)$, and then to infer results about the cohomology of $\widetilde{\sigma}(2)$.

Our method of computing the cohomology of $\overline{\sigma}(2)$, and comparing it with the cohomology of $\widetilde{\sigma}(2)$, will be to adapt a filtration employed by Ravenel to compute the cohomology of Morava stabilizer algebras. This filtration will result in a pair of May-type spectral sequences, which we refer to as *May-Ravenel* spectral sequences:

$$\begin{tikzcd}
{}^{MR}E_1(\widetilde{\sigma}(2)) = H^*(E_0^{MR}\widetilde{\sigma}(2)) \arrow[r, Rightarrow] \arrow[d] & H^*(\widetilde{\sigma}(2)) \arrow[d] \\
{}^{MR}E_1(\overline{\sigma}(2)) = H^*(E_0^{MR}\overline{\sigma}(2)) \arrow[r, Rightarrow] & H^*(\overline{\sigma}(2))
\end{tikzcd}$$

The E_1-terms ${}^{MR}E_1$ will be computed by endowing $E_0^{MR}\widetilde{\sigma}(2)$ and $E_0^{MR}\overline{\sigma}(2)$ with Adams filtrations, resulting in a pair of *Adams filtration* spectral sequences

$$\begin{tikzcd} {}^{AF}E_1(\widetilde{\sigma}(2)) \arrow[r, Rightarrow] \arrow[d] & H^*(E_0^{MR}\widetilde{\sigma}(2)) \arrow[d] \\ {}^{AF}E_1(\overline{\sigma}(2)) \arrow[r, Rightarrow] & H^*(E_0^{MR}\overline{\sigma}(2)) \end{tikzcd}$$

The May-Ravenel E_1-term ${}^{MR}E_1(\overline{\sigma}(2))$ is the cohomology of a certain restricted Lie algebra $\bar{l}(2)$. This cohomology may be computed by a Chevallay-Eilenberg complex, whose differentials were explicitly computed by Ravenel. The key observations which we employ are:

(1) The Chevallay-Eilenberg complex is isomorphic to ${}^{AF}E_1(\overline{\sigma}(2))$.
(2) The differentials in the Adams filtration spectral sequence $\{{}^{AF}E_r(\overline{\sigma}(2))\}$ can be completely computed in terms of the differentials in the Chevallay-Eilenberg complex.
(3) The image of ${}^{AF}E_1(\widetilde{\sigma}(2))$ in ${}^{AF}E_1(\overline{\sigma}(2))$ can be computed precisely, since we know the generators of $\widetilde{\sigma}(2)$ modulo terms of higher Adams filtration. This allows us to completely compute the differentials in the Adams filtration spectral sequence $\{{}^{AF}E_r(\widetilde{\sigma})\}$.

Even with knowing the differentials, the combinatorics for computing the spectral sequence $\{{}^{AF}E_r(\overline{\sigma})\}$ is complicated. This computation will be facilitated by refining the Adams filtration with a lexicographical filtration. This results in a *lexicographical filtration spectral sequence*

$${}^{AF}E_1(\overline{\sigma}(2)) = {}^{LF}E_0(\overline{\sigma}(2)) \Rightarrow {}^{MR}E_1(\overline{\sigma}(2)).$$

We will completely compute this spectral sequence, and thus completely compute ${}^{MR}E_1(\widetilde{\sigma}(2))$. In the low dimensional range we consider for our application, there will be no possible differentials in the May-Ravenel spectral sequence

$${}^{MR}E_1(\widetilde{\sigma}(2)) \Rightarrow H^*(\widetilde{\sigma}(2)).$$

5.2. **The May-Ravenel spectral sequence.** Let $(\mathbb{F}_2, S(2))$ be the Hopf algebra obtained from $(K(2)_*, \Sigma(2))$ by setting $v_2 = 1$. In [Rav86, Chapter 3], Ravenel computed

$$H^*(S(2)) = \mathrm{Ext}^*_{S(2)}(\mathbb{F}_2, \mathbb{F}_2).$$

The computation for $(K(2)_*, \Sigma(2))$ and $((K_2)_*, \Sigma_2)$ can be done using similar methods and all differentials follow from Ravenel's work by reintroducing the grading. We begin by summarizing Ravenel's method, which we then apply to our cases.

In [Rav86, Section 4.3], Ravenel defines a filtration of Hopf algebroids on $BP_*(BP)/I_N$. Specializing to the case of $N = p = 2$, this induces a filtration on $(k(2)_*, \sigma(2))$, where

$$\sigma(2) = \mathbb{F}_2[v_2][t_1, t_2, \ldots]/(t_k^4 - v_2^{2^k-1}t_k).$$

There is a unique increasing multiplicative filtration (which we call the *May-Ravenel filtration*) on $\sigma(2)$ such that

$$\begin{aligned} \deg(v_2) &= 0, \\ \deg(t_1^{2^j}) &= 1, \\ \deg(t_{2k+1}^{2^j}) &= 3 \cdot 2^{k-1}, \quad k > 0, \\ \deg(t_{2k}^{2^j}) &= 2^k. \end{aligned}$$

Further, Ravenel [Rav86, 4.3.24] proves that this is a filtration of Hopf algebroids, so that the associated graded $E_0(\sigma(2))$ is a Hopf algebra. It is given by the exterior algebra

$$E_0(\sigma(2)) \cong \mathbb{F}_2[v_2] \otimes E[t_{i,j} \;:\; 0 < i, j \in \{0,1\}]$$

where $t_{i,j}$ is the image of $t_i^{2^j}$.

From this filtration, we get a May type spectral sequence, which we call the *May-Ravenel spectral sequence*:

$${}^{MR}E_1(\sigma(2)) \Longrightarrow H^*(\sigma(2)).$$

The first step is to compute ${}^{MR}E_1(\sigma(2))$.

Let $E^0(\sigma(2))$ be the $\mathbb{F}_2$-linear dual of $E_0(\sigma(2))$ and $x_{i,j}$ be the dual of $t_{i,j}$. Since the $t_{i,j}$'s form a basis of the indecomposables of $E_0(\sigma(2))$, it follows that $x_{i,j}$ forms a basis for the restricted Lie algebra of primitives

$$l(2) := PE^0(\sigma(2))$$

and ${}^{MR}E_1 = H^*(l(2))$. Applying the methods of [May66, Remark 10], we obtain a Chevallay-Eilenberg cochain complex

$$C^*_{CE}(l(2)) := \mathbb{F}_2[v_2] \otimes P(\{h_{i,j}\}_{0<i, 0 \leq j \leq 1})$$

for elements $h_{i,j}$ of internal degree $2^{j+1}(2^i - 1)$ and cohomological degree 1 with

$$H^*(C^*_{CE}(l(2))) = {}^{MR}E_1(\sigma(2)).$$

(Here, $h_{i,j}$ represents the dual of the element May calls $\gamma_1(x_{i,j})$.)

The differentials are determined by the Lie bracket and restriction of $PE^0(\sigma(2))$. For $\sigma(2)$, these are obtained by "remembering the grading" in [Rav86, 6.3.3]. We obtain the following differentials.

Theorem 5.2.1. *Let $\chi_2 = v_2 h_{2,0} + h_{2,1}$. The differentials in $C^*_{CE}(l(2))$ are determined by $d(h_{1,0}) = d(h_{1,1}) = 0$ and*

$$\begin{aligned} d(h_{2,0}) &= h_{1,0}h_{1,1} & d(h_{2,1}) &= v_2 h_{1,0}h_{1,1} \\ d(h_{3,0}) &= h_{1,0}\chi_2 & d(h_{3,1}) &= v_2^2 h_{1,1}\chi_2 \\ d(h_{4,0}) &= h_{1,0}h_{3,1} + v_2^2 h_{1,1}h_{3,0} + v_2\chi_2^2 & d(h_{4,1}) &= v_2^5 h_{1,0}h_{3,1} + v_2^7 h_{1,1}h_{3,0} + v_2^6\chi_2^2 \\ d(h_{i,0}) &= v_2 h_{i-2,1}^2 & d(h_{i,1}) &= v_2^{2^{i-1}} h_{i-2,0}^2 \end{aligned}$$

where the last two identities hold for $i \geq 5$.

Now, we can put the same filtration on Σ_2, and this induces a filtration on $\overline{\Sigma}_2$ which restricts to a filtration on $\overline{\sigma}(2)$ and $\widetilde{\sigma}(2)$. The corresponding associated graded Hopf algebra in the case of $\overline{\sigma}(2)$ is given by

$$E_0^{MR}(\overline{\sigma}(2)) \cong \mathbb{F}_4[v_2] \otimes E(\widetilde{t}_{2,1}, \bar{t}_{3,0}, \bar{t}_{3,1}, \bar{t}_{4,0}, \bar{t}_{4,1}, \cdots).$$

As before, we have a May-Ravenel spectral sequence

$${}^{MR}E_1(\overline{\sigma}(2)) \Longrightarrow H^*(\overline{\sigma}(2)),$$

and ${}^{MR}E_1(\overline{\sigma}(2)) = H^*(\bar{l}(2))$ is computed via a Chevallay-Eilenberg complex

$$C^*_{CE}(\bar{l}(2)) \cong \mathbb{F}_4[v_2, \widetilde{h}_{2,1}, h_{3,0}, h_{3,1}, h_{4,0}, h_{4,1}, \ldots].$$

The following is then a consequence of Theorem 5.2.1, and the fact that χ_2 corresponds to $\widetilde{h}_{2,1}$ in the Chevellay-Eilenberg complex $C^*_{CE}(\bar{l}(2))$ (see (4.4.7)).

Theorem 5.2.2. *The differentials in the Chevallay-Eilenberg complex $C^*_{CE}(\bar{l}(2))$ are determined by*

$$d(\widetilde{h}_{2,1}) = d(h_{3,0}) = d(h_{3,1}) = 0$$

and

$$\begin{aligned} d(h_{4,0}) &= v_2 \widetilde{h}^2_{2,1} & \qquad d(h_{4,1}) &= v_2^6 \widetilde{h}^2_{2,1} \\ d(h_{i,0}) &= v_2 h^2_{i-2,1} & \qquad d(h_{i,1}) &= v_2^{2^{i-1}} h^2_{i-2,0} \end{aligned}$$

where the last two identities hold for $i \geq 5$.

5.3. **The lexicographical filtration spectral sequence.** In order to compute the cohomology of this Chevallay-Eilenberg complex $C^*_{CE}(\bar{l}(2))$, we order the monomials via a lexicographical filtration. Express a generic additive generator of $C^*_{CE}(\bar{l}(2))$ in the form

$$v_2^m h_{3,0}^{k_3} h_{4,0}^{k_4} \cdots \widetilde{h}_{2,1}^{l_2} h_{3,1}^{l_3} \widetilde{h}_{4,1}^{l_4} \cdots .$$

We place an increasing filtration on these monomials via left lexicographical[5] ordering on the sequence

$$(-m; \cdots, \bar{l}_6, \bar{l}_5; \cdots \bar{k}_5, \bar{k}_4)$$

where in the above $\bar{n} \in \{0, 1\}$ is n mod 2. Note that the value m above is the Adams filtration, so lexicographical filtration is a refinement of Adams filtration. The differentials of Theorem 5.2.2 are easily seen to decrease lexicographical ordering, resulting in an increasing filtration on $C^*_{CE}(\bar{l}(2))$ and a (transfinite) lexicographical filtration spectral sequence (LFSS)

$$C^*_{CE}(\bar{l}(2)) = {}^{LF}E_0 \Rightarrow {}^{MR}E_1(\overline{\sigma}(2)).$$

We decline to explicitly grade this spectral sequence, and will simply remember that differentials decrease lexicographical filtration, and that they are ordered by which differential changes lexicographical filtration by the least amount. Note that differentials increase Adams filtration.

[5] We remind the reader that left lexicographical filtration imposes an order on a multi-index by first comparing the left-most (first) index, but in the case of equality compares the second index, and so on.

We now run this spectral sequence. We will run the differentials in two rounds. The first round will consist of those differentials which change Adams filtration by 1. The second round will consist of those differentials which change Adams filration by a quantity greater than 1.

Differentials which change Adams filtration by 1*.*

The first round of differentials are

$$d(v_2^m h_{3,0}^{k_3} h_{4,0}^{2k_4+1} h_{5,0}^{k_5} \cdots \widetilde{h}_{2,1}^{l_2} h_{3,1}^{l_3} h_{4,1}^{l_4} \cdots) = v_2^{m+1} h_{3,0}^{k_3} h_{4,0}^{2k_4} h_{5,0}^{k_5} \cdots \widetilde{h}_{2,1}^{l_2+2} h_{3,1}^{l_3} \widetilde{h}_{4,1}^{l_4} \cdots .$$

What remains has basis

$$\begin{gathered} h_{3,0}^{k_3} h_{4,0}^{2k_4} h_{5,0}^{k_5} \cdots \widetilde{h}_{2,1}^{l_2+2} h_{3,1}^{l_3} \widetilde{h}_{4,1}^{l_4} \cdots , \\ v_2^m h_{3,0}^{k_3} h_{4,0}^{2k_4} h_{5,0}^{k_5} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{l_3} \widetilde{h}_{4,1}^{l_4} \cdots . \end{gathered}$$

Here, $\epsilon_2 \in \{0,1\}$. The next round of differentials are those of the form

$$d(v_2^m h_{3,0}^{k_3} h_{4,0}^{2k_4} h_{5,0}^{2k_5+1} h_{6,0}^{k_6} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{l_3} \widetilde{h}_{4,1}^{l_4} \cdots) = v_2^{m+1} h_{3,0}^{k_3} h_{4,0}^{2k_4} h_{5,0}^{2k_5} h_{6,0}^{k_6} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{l_3+2} \widetilde{h}_{4,1}^{l_4} \cdots .$$

What remains has basis

$$\begin{gathered} h_{3,0}^{k_3} h_{4,0}^{2k_4} h_{5,0}^{k_5} \cdots \widetilde{h}_{2,1}^{l_2+2} h_{3,1}^{l_3} \widetilde{h}_{4,1}^{l_4} \cdots , \\ h_{3,0}^{k_3} h_{4,0}^{2k_4} h_{5,0}^{2k_5} h_{6,0}^{k_6} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{l_3+2} \widetilde{h}_{4,1}^{l_4} \cdots , \\ v_2^m h_{3,0}^{k_3} h_{4,0}^{2k_4} h_{5,0}^{2k_5} h_{6,0}^{k_6} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{l_4} \cdots \end{gathered}$$

with $\epsilon_3 \in \{0,1\}$. Repeating this process infinitely many times, we have the following.

Lemma 5.3.1. *The page of the lexicographical filtration spectral sequence obtained by running all differentials which increase Adams filtration by* 1 *has a basis whose leading terms (with respect to lexicographical ordering) are given by:*

$$\text{(I)} \quad v_2^m h_{3,0}^{k_3} h_{4,0}^{2k_4} h_{5,0}^{2k_5} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \cdots , \qquad m, k_j \geq 0;\ \epsilon_j \in \{0,1\},$$

$$\text{(II)} \quad h_{3,0}^{k_3} h_{4,0}^{2k_4} \cdots h_{i+2,0}^{2k_{i+2}} h_{i+3,0}^{k_{i+3}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i+2} h_{i+1,1}^{l_{i+1}} \cdots , \qquad i \geq 2;\ k_j, l_j \geq 0;\ \epsilon_j \in \{0,1\}.$$

Proof. The only extra thing to check is that there are no additional differentials. The subtle point is that differentials cannot be computed just by considering the leading terms. We therefore must argue that terms (I) and (II) can be completed, by adding terms of lower lexicographical filtration, to cocycles (with respect to the differentials which raise Adams filtration by 1). In the case of terms (I), this is trivially true - the leading terms are cocycles (with respect to differentials which change Adams filtration 1). In the case of terms of type (II), one can check that the sum (with $\bar{\epsilon}_j \in \{0,1\}$)

$$
\begin{aligned}
&x(k_3, 2k_4, \dots 2k_{i+2}, 2k_{i+3} + \bar{\epsilon}_{i+3}, \dots; \epsilon_2, \dots \epsilon_{i-1}, l_i + 2, l_{i+1}, \dots) := \\
&h_{3,0}^{k_3} h_{4,0}^{2k_4} \cdots h_{i+2,0}^{2k_{i+2}} h_{i+3,0}^{2k_{i+3}+\bar{\epsilon}_{i+3}} h_{i+4,0}^{2k_{i+4}+\bar{\epsilon}_{i+4}} h_{i+5,0}^{2k_{i+5}+\bar{\epsilon}_{i+5}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i+2} h_{i+1,1}^{l_{i+1}} \cdots \\
&+ \bar{\epsilon}_{i+3} h_{3,0}^{k_3} h_{4,0}^{2k_4} \cdots h_{i+2,0}^{2k_{i+2}+1} h_{i+3,0}^{2k_{i+3}} h_{i+4,0}^{2k_{i+4}+\bar{\epsilon}_{i+4}} h_{i+5,0}^{2k_{i+5}+\bar{\epsilon}_{i+5}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i} h_{i+1,1}^{l_{i+1}+2} h_{i+2,1}^{l_{i+2}} \cdots \\
&+ \bar{\epsilon}_{i+4} h_{3,0}^{k_3} h_{4,0}^{2k_4} \cdots h_{i+2,0}^{2k_{i+2}+1} h_{i+3,0}^{2k_{i+3}+\bar{\epsilon}_{i+3}} h_{i+4,0}^{2k_{i+4}} h_{i+5,0}^{2k_{i+5}+\bar{\epsilon}_{i+5}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i} h_{i+1,1}^{l_{i+1}} h_{i+2,1}^{l_{i+2}+2} \cdots \\
&+ \bar{\epsilon}_{i+5} h_{3,0}^{k_3} h_{4,0}^{2k_4} \cdots h_{i+2,0}^{2k_{i+2}+1} h_{i+3,0}^{2k_{i+3}+\bar{\epsilon}_{i+3}} h_{i+4,0}^{2k_{i+4}+\bar{\epsilon}_{i+4}} h_{i+5,0}^{2k_{i+5}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i} \cdots h_{i+2,1}^{l_{i+2}} h_{i+3,1}^{l_{i+3}+2} \cdots \\
&\qquad + \cdots
\end{aligned}
$$

is such a cocycle. ☐

Differentials which change Adams filtration by more than 1.

We now run the differentials which change Adams filtration by more than 1. The idea is that these differentials are non-trivial only on terms of type (I), and these differentials hit terms of type (I). Terms of type (II) are going to be permanent cycles in the lexicographical filtration spectral sequence.

We note in what follows that by Theorem 5.2.2, the element $h_{4,1}$ is a permanent cycle in the LFSS. This is because the target of $d(h_{4,1})$ is killed by the shorter differential

$$d(v_2^5 h_{4,0}) = v_2^6 \widetilde{h}_{2,1}^2.$$

The first round of differentials in the LFSS will be of the form

$$
\begin{aligned}
&d(v_2^m h_{3,0}^{\bar{\epsilon}_3} h_{3,0}^{2k_3} h_{4,0}^{2k_4} h_{5,0}^{2k_5} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} h_{5,1} h_{6,1}^{\epsilon_6} \cdots) \\
&\qquad\qquad = v_2^{m+16} h_{3,0}^{\bar{\epsilon}_3} h_{3,0}^{2(k_3+1)} h_{4,0}^{2k_4} h_{5,0}^{2k_5} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} h_{6,1}^{\epsilon_6} \cdots
\end{aligned}
$$

with $m, k_j \in \mathbb{N}$ and $\epsilon_j, \bar{\epsilon}_j \in \{0,1\}$. Of the terms of type (I), what remains are terms of the forms

$$
\begin{gathered}
v_2^m h_{3,0}^{\bar{\epsilon}_3} h_{4,0}^{2k_4} h_{5,0}^{2k_5} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} h_{6,1}^{\epsilon_6} \cdots, \\
v_2^{<16} h_{3,0}^{\bar{\epsilon}_3} h_{3,0}^{2(k_3+1)} h_{4,0}^{2k_4} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} h_{6,1}^{\epsilon_6} \cdots.
\end{gathered}
$$

The next round of differentials will be of the form

$$
\begin{aligned}
&d(v_2^m h_{3,0}^{\bar{\epsilon}_3} h_{4,0}^{2k_4} h_{5,0}^{2k_5} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} h_{6,1} h_{7,1}^{\epsilon_7} \cdots) \\
&\qquad\qquad = v_2^{m+32} h_{3,0}^{\bar{\epsilon}_3} h_{4,0}^{2(k_4+1)} h_{5,0}^{2k_5} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} h_{7,1}^{\epsilon_7} \cdots.
\end{aligned}
$$

Of the terms of type (I), what remain are terms of the forms

$$
\begin{gathered}
v_2^m h_{3,0}^{\bar{\epsilon}_3} h_{5,0}^{2k_5} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} h_{7,1}^{\epsilon_7} \cdots, \\
v_2^{<16} h_{3,0}^{\bar{\epsilon}_3} h_{3,0}^{2(k_3+1)} h_{4,0}^{2k_4} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} h_{6,1}^{\epsilon_6} \cdots, \\
v_2^{<32} h_{3,0}^{\bar{\epsilon}_3} h_{4,0}^{2(k_4+1)} h_{5,0}^{2k_5} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} h_{7,1}^{\epsilon_7} \cdots.
\end{gathered}
$$

Continuing in this manner, we arrive at the following.

Theorem 5.3.2. *The May-Ravenel E_1-term ${}^{MR}E_1(\overline{\sigma}(2))$ has a basis over $\mathbb{F}_4$ whose representatives in the lexicographical filtration spectral sequence are given by:*

$$(\mathrm{I}')\quad v_2^m h_{3,0}^{\bar\epsilon_3}\widetilde{h}_{2,1}^{\epsilon_2}h_{3,1}^{\epsilon_3}\widetilde{h}_{4,1}^{\epsilon_4},$$

$$m \geq 0;\ \epsilon_j, \bar\epsilon_j \in \{0,1\},$$

$$(\mathrm{I}'')\quad v_2^{<2^{i+1}} h_{3,0}^{\bar\epsilon_3} h_{i,0}^{2(k_i+1)} h_{i+1,0}^{2k_{i+1}} h_{i+2,0}^{2k_{i+2}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3}\widetilde{h}_{4,1}^{\epsilon_4} h_{i+3,1}^{\epsilon_{i+3}} \cdots .$$

$$i \geq 3;\ k_j \geq 0;\ \epsilon_j, \bar\epsilon_j \in \{0,1\},$$

$$(\mathrm{II})\quad h_{3,0}^{k_3} h_{4,0}^{2k_4} \cdots h_{i+2,0}^{2k_{i+2}} h_{i+3,0}^{k_{i+3}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i+2} h_{i+1,1}^{l_{i+1}} \cdots .$$

$$i \geq 2;\ k_j, l_j \geq 0;\ \epsilon_j \in \{0,1\}.$$

Proof. As with Lemma 5.3.1, the only missing piece of the analysis above is a proof that the listed terms are indeed permanent cycles in the lexicographical filtration spectral sequence. This again will be accomplished by explicitly exhibiting cocycles in $C^*_{CE}(\bar{l}(2))$ with leading terms agreeing with those above. The terms (I$'$) are simply cocycles. The terms (I$''$) complete to cocycles given by

$$\begin{aligned}
&h_{3,0}^{\bar\epsilon_3} h_{i,0}^{2(k_i+1)} h_{i+1,0}^{2k_{i+1}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3}\widetilde{h}_{4,1}^{\epsilon_4} h_{i+3,1}^{\epsilon_{i+3}} \cdots \\
&\quad + \epsilon_{i+3} v_2^{2^{i+2}-2^{i+1}} h_{3,0}^{\bar\epsilon_3} h_{i,0}^{2k_i} h_{i+1,0}^{2(k_{i+1}+1)} h_{i+2,0}^{2k_{i+2}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3}\widetilde{h}_{4,1}^{\epsilon_4} h_{i+2,1} h_{i+4,1}^{\epsilon_{i+4}} \cdots \\
&\ + \epsilon_{i+4} v_2^{2^{i+3}-2^{i+1}} h_{3,0}^{\bar\epsilon_3} h_{i,0}^{2k_i} h_{i+1,0}^{2k_{i+1}} h_{i+2,0}^{2(k_{i+2}+1)} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3}\widetilde{h}_{4,1}^{\epsilon_4} h_{i+2,1} h_{i+3,1}^{\epsilon_{i+3}} h_{i+5,1}^{\epsilon_{i+5}} \cdots \\
&\qquad + \cdots
\end{aligned}$$

For the terms of type (II), we observe that the Cartan-Eilenberg differential d^{CE} is given on the terms $x(-)$ appearing in the proof of Lemma 5.3.1 by

$$\begin{aligned}
&d^{CE} x(k_3, 2k_4, \ldots 2k_{i+2}, 2k_{i+3}+\bar\epsilon_{i+3}, \ldots; \epsilon_2, \ldots \epsilon_{i-1}, l_i+2, l_{i+1}, \ldots) = \\
&\epsilon_5 v_2^{2^4} x(k_3+2, 2k_4, \ldots 2k_{i+2}, 2k_{i+3}+\bar\epsilon_{i+3}, \ldots; \epsilon_2, \ldots, \epsilon_4, 0, \epsilon_6, \ldots, \epsilon_{i-1}, l_i+2, l_{i+1}, \ldots) \\
&+\epsilon_6 v_2^{2^5} x(k_3, 2(k_4+1), \ldots 2k_{i+2}, 2k_{i+3}+\bar\epsilon_{i+3}, \ldots; \epsilon_2, \ldots, \epsilon_4, \epsilon_5, 0, \epsilon_7, \ldots, \epsilon_{i-1}, l_i+2, l_{i+1}, \ldots) \\
&\qquad + \cdots \\
&+\bar{l}_i v_2^{2^{i-1}} x(k_3, 2k_4, \ldots, 2(k_{i-2}+1), \ldots, 2k_{i+2}, 2k_{i+3}+\bar\epsilon_{i+3}, \ldots; \epsilon_2, \ldots \epsilon_{i-1}, l_i-1+2, l_{i+1}, \ldots) \\
&+\bar{l}_{i+1} v_2^{2^i} x(k_3, 2k_4, \ldots, 2(k_{i-1}+1), \ldots, 2k_{i+2}, 2k_{i+3}+\bar\epsilon_{i+3}, \ldots; \epsilon_2, \ldots \epsilon_{i-1}, l_i+2, l_{i+1}-1, \ldots) \\
&\qquad + \cdots .
\end{aligned}$$

However, also note that

$$\begin{aligned}
&d^{CE}(h_{3,0}^{k_3} h_{4,0}^{2k_4} \cdots h_{i+2,0}^{2k_{i+2}+1} h_{i+3,0}^{2k_{i+3}+\bar\epsilon_{i+3}} h_{i+4,0}^{2k_{i+4}+\bar\epsilon_{i+4}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i} h_{i+1,1}^{l_{i+1}} \cdots \\
&\qquad = v_2 x(k_3, 2k_4, \ldots, 2k_{i+2}, 2k_{i+3}+\bar\epsilon_{i+3}, \ldots; \epsilon_2, \ldots \epsilon_{i-1}, l_i+2, l_{i+1}, \ldots).
\end{aligned}$$

We therefore find that the terms of type (II) complete to the following cocycles:

$$
\begin{gathered}
x(k_3, 2k_4, \dots 2k_{i+2}, 2k_{i+3} + \bar{\epsilon}_{i+3}, \dots; \epsilon_2, \dots \epsilon_{i-1}, l_i + 2, l_{i+1}, \dots) \\
+ \epsilon_5 v_2^{2^4-1} h_{3,0}^{k_3+2} h_{4,0}^{2k_4} \cdots h_{i+2,0}^{2k_{i+2}+1} h_{i+3,0}^{2k_{i+3}+\bar{\epsilon}_{i+3}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots \widetilde{h}_{4,1}^{\epsilon_4} h_{6,1}^{\epsilon_6} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i} \cdots \\
+ \epsilon_6 v_2^{2^5-1} h_{3,0}^{k_3} h_{4,0}^{2(k_4+1)} \cdots h_{i+2,0}^{2k_{i+2}+1} h_{i+3,0}^{2k_{i+3}+\bar{\epsilon}_{i+3}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots \widetilde{h}_{4,1}^{\epsilon_4} h_{5,1}^{\epsilon_5} h_{7,1}^{\epsilon_7} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i} \cdots \\
+ \cdots \\
+\bar{l}_i v_2^{2^{i-1}-1} h_{3,0}^{k_3} h_{4,0}^{2k_4} \cdots h_{i-2,0}^{2(k_{i-2}+1)} \cdots h_{i+2,0}^{2k_{i+2}+1} h_{i+3,0}^{2k_{i+3}+\bar{\epsilon}_{i+3}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots \widetilde{h}_{4,1}^{\epsilon_4} h_{5,1}^{\epsilon_5} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i-1} \cdots \\
+\bar{l}_{i+1} v_2^{2^{i}-1} h_{3,0}^{k_3} h_{4,0}^{2k_4} \cdots h_{i-2,0}^{2(k_{i-1}+1)} \cdots h_{i+2,0}^{2k_{i+2}+1} h_{i+3,0}^{2k_{i+3}+\bar{\epsilon}_{i+3}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots \widetilde{h}_{4,1}^{\epsilon_4} h_{5,1}^{\epsilon_5} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i} h_{i+1,1}^{l_{i+1}-1} \cdots \\
+ \cdots \\
+\bar{l}_{i+4} v_2^{2^{i+3}-1} h_{3,0}^{k_3} h_{4,0}^{2k_4} \cdots h_{i+2,0}^{2(k_{i+2}+1)+1} h_{i+3,0}^{2k_{i+3}+\bar{\epsilon}_{i+3}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots \widetilde{h}_{4,1}^{\epsilon_4} h_{5,1}^{\epsilon_5} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i} \cdots h_{i+4,1}^{l_{i+4}-1} \cdots \\
+\bar{l}_{i+5} v_2^{2^{i+4}-1} h_{3,0}^{k_3} h_{4,0}^{2k_4} \cdots h_{i+2,0}^{2k_{i+2}+1} h_{i+3,0}^{2(k_{i+3}+1)+\bar{\epsilon}_{i+3}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots \widetilde{h}_{4,1}^{\epsilon_4} h_{6,1}^{\epsilon_6} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i} \cdots h_{i+5,1}^{l_{i+5}-1} \cdots \\
+ \cdots .
\end{gathered}
$$

□

5.4. **The Adams filtration spectral sequence.** Endow $\overline{\sigma}(2)$ and its subalgebra $\widetilde{\sigma}(2)$ with an increasing Adams filtration, by declaring $AF(v_2) = 1$, and giving all other generators Adams filtration 0. The differentials in the cobar complex respect Adams filtration because, by Theorem 4.4.5, they come from maps of spectra (the connecting maps in the tmf-Adams resolution for Z).

The algebra generators of

$$E_0^{AF}\overline{\sigma}(2) = \mathbb{F}_4[v_2, \widetilde{t_2^2}, \bar{t}_3, \bar{t}_4, \dots]/(\widetilde{t_2^2} = 0, \bar{t}_k^4 = 0)$$

are easily seen to be primitive (see, for example, [Rav92a, Prop. B.5.15]). Furthermore, Theorem 4.4.5 implies that $E_0^{AF}\widetilde{\sigma}(2)$ is the primitively generated $k(2)_*$-subalgebra

$$\mathbb{F}_2[v_2, \widetilde{t_2^2}, \bar{t}_3, \bar{t}_4, \dots]/(\widetilde{t_2^2} = 0, \bar{t}_k^4 = 0) \subset E_0^{AF}\overline{\sigma}(2).$$

Remark 5.4.1. Since there is an isomorphism of cochain complexes

$$\mathcal{C}_{alg}^{*,*,*}(Z) \cong C^*_{E_0^{AF}\widetilde{\sigma}(2)}(k(2)_*)$$

we deduce

$$H^{*,*,*}(\mathcal{C}_{alg}) \cong \mathbb{F}_2[v_2, \widetilde{h}_{2,1}, h_{i,j}]_{\substack{i\geq 3 \\ j=0,1}}.$$

We may likewise endow $E_0^{MR}\overline{\sigma}(2)$ and $E^{MR}\widetilde{\sigma}(2)$ with Adams filtration. Then $E_0^{AF}E_0^{MR}\overline{\sigma}(2)$ is given by

$$\mathbb{F}_4[v_2] \otimes E(\widetilde{t}_{2,1}, \bar{t}_{i,j})_{\substack{i\geq 3 \\ j=0,1}}$$

with $\widetilde{t}_{2,1}$, $t_{i,j}$ primitive, and $E_0^{AF}E_0^{MR}\overline{\sigma}(2)$ is given by the subalgebra

$$\mathbb{F}_4[v_2] \otimes E(\widetilde{t}_{2,1}, \bar{t}_{i,j})_{\substack{i\geq 3 \\ j=0,1}}.$$

This results in a pair of *Adams filtration* spectral sequences

$$\begin{tikzcd} {}^{AF}E_1(\widetilde{\sigma}(2)) \arrow[r, Rightarrow] \arrow[d] & H^*(E_0^{MR}\widetilde{\sigma}(2)) \arrow[d] \\ {}^{AF}E_1(\overline{\sigma}(2)) \arrow[r, Rightarrow] & H^*(E_0^{MR}\overline{\sigma}(2)) \end{tikzcd}$$

with

$$\begin{aligned} {}^{AF}E_1(\widetilde{\sigma}(2)) &\cong \mathbb{F}_2[v_2, \widetilde{h}_{2,1}, h_{i,j}]_{\substack{i\geq 3\\ j=0,1}}, \\ {}^{AF}E_1(\overline{\sigma}(2)) &\cong \mathbb{F}_4[v_2, \widetilde{h}_{2,1}, h_{i,j}]_{\substack{i\geq 3\\ j=0,1}}. \end{aligned}$$

We will now compute the Adams filtration spectral sequence ${}^{AF}E_r(\overline{\sigma}(2))$ by relating it to the LFSS. Namely, the Chevallay-Eilenberg complex $C^*_{CE}(\bar{l}(2))$ is a quotient of the cobar complex for $E_0^{MR}\overline{\sigma}(2)$

$$C^*_{E_0^{MR}\overline{\sigma}(2)} \twoheadrightarrow C^*_{CE}(\bar{l}(2)).$$

By endowing $C^*_{CE}(\bar{l}(2))$ with an Adams filtration, we get an associated spectral sequence ${}^{AF}E_r(\bar{l}(2))$ and a map of spectral sequences

$$\begin{tikzcd} {}^{AF}E_1(\overline{\sigma}(2)) \arrow[r, Rightarrow] \arrow[d] & H^*(E_0^{MR}\overline{\sigma}(2)) \arrow[d, equal] \\ {}^{AF}E_1(\bar{l}(2)) \arrow[r, Rightarrow] & H^*(E_0^{MR}\overline{\sigma}(2)) \end{tikzcd}$$

From Theorem 5.2.2 we see that all differentials in $C^*_{CE}(\bar{l}(2))$ increase Adams filtration, and thus

$$\begin{aligned} {}^{AF}E_1(\bar{l}(2)) &= H^*(E_0^{AF}C^*_{CE}(\bar{l}(2))) \\ &= E_0^{AF}C^*_{CE}(\bar{l}(2)) \\ &= \mathbb{F}_4[v_2, \widetilde{h}_{2,1}, h_{i,j}]_{\substack{i\geq 3\\ j=0,1}}. \end{aligned}$$

We deduce the following.

Proposition 5.4.2. *The map*

$${}^{AF}E_1(\overline{\sigma}(2)) \to {}^{AF}E_1(\bar{l}(2))$$

is an isomorphism, and thus there is an isomorphism of spectral sequences

$$\{{}^{AF}E_r(\overline{\sigma}(2))\} \cong \{{}^{AF}E_r(\bar{l}(2))\}.$$

Finally, we observe that since lexicographic filtration is a refinement of Adams filtration, the differentials in the AFSS ${}^{AF}E_r(\bar{l}(2))$ are simply those differentials in the LFSS which change Adams filtration by r. As we have determined the LFSS, we have implicitely determined the AFSS for $\bar{l}(2)$, and therefore the AFSS for $\overline{\sigma}(2)$. We deduce that the AFSS for $\overline{\sigma}(2)$ is simply obtained by restricting the differentials from the AFSS for $\overline{\sigma}(2)$. Therefore, from Theorem 5.3.2 we deduce:

Theorem 5.4.3. *The May-Ravenel E_1-term ${}^{MR}E_1(\widetilde{\sigma}(2))$ has a basis over $\mathbb{F}_2$ whose representatives in the lexicographical filtration spectral sequence are given by:*

$$(\mathrm{I}') \quad v_2^m h_{3,0}^{\bar{\epsilon}_3} \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4}, \qquad m \geq 0;\ \epsilon_j, \bar{\epsilon}_j \in \{0,1\},$$

$$(\mathrm{I}'') \quad v_2^{<2^{i+1}} h_{3,0}^{\bar{\epsilon}_3} h_{i,0}^{2(k_i+1)} h_{i+1,0}^{2k_{i+1}} h_{i+2,0}^{2k_{i+2}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} h_{i+3,1}^{\epsilon_{i+3}} \cdots . \qquad i \geq 3;\ k_j \geq 0;\ \epsilon_j, \bar{\epsilon}_j \in \{0,1\},$$

$$(\mathrm{II}) \quad h_{3,0}^{k_3} h_{4,0}^{2k_4} \cdots h_{i+2,0}^{2k_{i+2}} h_{i+3,0}^{k_{i+3}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i+2} h_{i+1,1}^{l_{i+1}} \cdots . \qquad i \geq 2;\ k_j, l_j \geq 0;\ \epsilon_j \in \{0,1\}.$$

Remark 5.4.4. We do not know if there are differentials in the May-Ravenel spectral sequence
$$ {}^{MR}E_1(\widetilde{\sigma}(2)) \Rightarrow H^*(\widetilde{\sigma}(2)) \cong H^{*,*}(\mathcal{C}). $$
Even in relatively low degrees, possibilities are plentiful. For example, there could be a differential
$$ d_4^{MR}(h_{5,0}^2) \overset{?}{=} v_2^{14} \widetilde{h}_{2,1} h_{3,0}^2. $$
We also do not know if there are possible hidden v_2-extensions in the May-Ravenel spectral sequence. Again, there are endless possibilities - as an example, there could be a hidden extension
$$ v_2^{16} h_{3,0}^4 \overset{?}{=} v_2^{14} h_{2,1} h_{3,0}^2 h_{3,1}. $$
However, in the very low degrees which are relevant to the computations later in this paper, there are no possibilities of differentials or hidden v_2-extensions.

6. The agathokakological method

In this section we will adapt the agathokakological method introduced in [BBB+17] to our present setting, to compute the E_2-term of the tmf-ASS for Z.

6.1. **Overview of the method.**

The short exact sequences
$$ 0 \to V^{*,*}(Z) \to {}^{\mathrm{tmf}}E_1^{*,*}(Z) \to \mathcal{C}^{*,*}(Z) \to 0., $$
$$ 0 \to V_{alg}^{*,*,*}(Z) \to {}^{\mathrm{tmf}}_{alg}E_1^{*,*,*} \to \mathcal{C}_{alg}^{*,*,*}(Z) \to 0 $$
give rise to long exact sequences
$$ \cdots \to H^{*,*}(V) \to {}^{\mathrm{tmf}}E_2^{*,*}(Z) \to H^{*,*}(\mathcal{C}) \xrightarrow{\partial} H^{*+1,*}(V) \to \cdots , $$
$$ \cdots \to H^{*,*,*}(V_{alg}) \to {}^{\mathrm{tmf}}_{alg}E_2^{*,*,*}(Z) \to H^{*,*,*}(\mathcal{C}_{alg}) \xrightarrow{\partial_{alg}} H^{*+1,*,*}(V_{alg}) \to \cdots $$
We have computed $H^{*,*,*}(\mathcal{C}_{alg})$ (Remark 5.4.1). By using Bruner's Ext program to compute ${}^{ass}E_2(Z)$ through a range, we can then use the Mahowald spectral sequence
$$ {}^{\mathrm{tmf}}_{alg}E_2(Z) \Rightarrow {}^{ass}E_2(Z) $$

to deduce $H^{*,*}(V)$. We then use our knowledge of $H^{*,*}(\mathcal{C})$ (Theorem 5.4.3 and Remark 5.4.4) to deduce ${}^{\mathrm{tmf}}E_2(Z)$.

This analysis is aided by means of the *(algebraic) agathokakological spectral sequence (AKSS)*.

$$\{{}^{akss}_{alg}E^{n+\alpha\epsilon,s,t}_{r+\beta\epsilon}(Z)\} \Rightarrow {}^{ass}E^{n+s,t}_2(Z)$$

with

$$\begin{aligned} n, s, t &\in \mathbb{N}, \\ \alpha &\in \{0,1\}, \\ \beta &\in \{-1,0,1\} \end{aligned}$$

The pages of this spectral sequence are ordered by

$$n-\epsilon < n < n+\epsilon < n+1$$

and the differentials take the form

$$\begin{aligned} d^{akss}_{r-\epsilon} &: {}^{akss}_{alg}E^{n+\epsilon,s,t}_{r-\epsilon} \to {}^{akss}_{alg}E^{n+r,s-r+1,t}_{r-\epsilon}, \\ d^{akss}_{r} &: {}^{akss}_{alg}E^{n+\alpha\epsilon,s,t}_{r} \to {}^{akss}_{alg}E^{n+r+\alpha\epsilon,s-r+1,t}_{r}, \\ d^{akss}_{r+\epsilon} &: {}^{akss}_{alg}E^{n,s,t}_{r+\epsilon} \to {}^{akss}_{alg}E^{n+r+\epsilon,s-r+1,t}_{r+\epsilon}. \end{aligned}$$

We have

$${}^{akss}_{alg}E^{n+\alpha\epsilon,s,t}_{1+\epsilon}(Z) = \begin{cases} H^{n,s,t}(\mathcal{C}_{alg}), & \alpha = 0, \\ H^{n,t}(V), & \alpha = 1, s = 0, \\ 0, & \text{otherwise} \end{cases}$$

and

$$d^{akss}_{1+\epsilon} = \partial_{alg}.$$

See [BBB+17, Sec. 7] for a detailed account of the construction of this spectral sequence.

Elements in ${}^{akss}_{alg}E^{n,s,t}_{r+\beta\epsilon}(Z)$ are called *good*, and elements in ${}^{akss}_{alg}E^{n+\epsilon,s,t}_{r+\beta\epsilon}(Z)$ are called *evil*. Non-trivial elements of ${}^{ass}E_2(Z)$ are called *good* (respectively *evil*) if they are detected in the AKSS by good (respectively evil) classes. Because there are no evil classes in tridegrees with $s > 0$, there are no non-trivial differentials

$$d_{r+\beta\epsilon}(x) = y$$

with x evil and $r > 1$.

6.2. **The dichotomy principle.** The key to computing the algebraic AKSS is to determine which elements of ${}^{ass}E_2(Z)$ are good and which are evil. This is done by linking v_2-periodicity with goodness. An element of ${}^{ass}E_2(Z)$ is *v_2-periodic* if its image under the homomorphism

$${}^{ass}E_2(Z) \to v_2^{-1}\,{}^{ass}E_2(Z)$$

is non-trivial. Otherwise it is said to be *v_2-torsion*.

The following two propositions give a practical means of determining whether an element of ${}^{ass}E_2(Z)$ is v_2-periodic.

Proposition 6.2.1. *We have*

$$v_2^{-1}\,{}^{ass}E_2(Z) \cong \mathbb{F}_2[v_2^{\pm}, \widetilde{h}_{2,1}, h_{3,0}, h_{3,1}, h_{4,0}, h_{4,1}, \cdots].$$

Proof. The computation is almost identical to that of [MRS01, (2.20)]. □

Corollary 6.2.2. *For $r > 1$, there are no d_r differentials between good classes in the algebraic AKSS.*

Proof. Proposition 6.2.1 implies that the v_2-localized algebraic AKSS collapses at $E_{1+\epsilon}$. The result follows from the fact that the map

$${}^{akss}_{alg}E_{1+\epsilon}^{n+\alpha\epsilon,s,t}(Z) \hookrightarrow v_2^{-1}\,{}^{akss}_{alg}E_{1+\epsilon}^{n+\alpha\epsilon,s,t}(Z)$$

is an injection for $\alpha = 0$ (the good part). □

In order to state and prove the dichotomy principle, we will need to establish bounds on v_2-periodicity in Ext, and on the evil complex. Let A_2 denote the cofiber of the v_2-self map

$$\Sigma^6 Z \to Z.$$

We have

$$H^*(A_2) \cong A(2)$$

as an $A(2)$-module (see Section 7.1).

Lemma 6.2.3. *We have*

$${}^{ass}E_2^{s,t}(A_2) = 0$$

for

$$s > \frac{(t-s)+12}{11}.$$

Proof. The May spectral sequence for ${}^{ass}E_2(A_2 \wedge C\sigma)$ has E_1-term of the form[6]

$${}^{May}E_1^{*,*,*}(A_2 \wedge C\sigma) \cong \mathbb{F}_2[h_{1,j_1}, h_{2,j_2}, h_{3,j_3}, \cdots : j_1 \geq 4; j_2 \geq 2; j_3 \geq 1; j_k \geq 0, k \geq 4].$$

One checks that the smallest slope $\frac{s}{t-s}$ of these generators is $\frac{1}{11}$, given by $h_{2,2}$. It follows that

$${}^{ass}E_2^{s,t}(A_2 \wedge C\sigma) = 0$$

for

$$s > \frac{t-s}{11}.$$

It follows from the fact that $h_{1,3}^4 = 0$ in ${}^{ass}E_2^{*,*}(S)$ that the $h_{1,3}$-Bockstein spectral sequence

$${}^{ass}E_2^{*,*}(A_2 \wedge C\sigma)[h_{1,3}] \Rightarrow {}^{ass}E_2^{*,*}(A_2)$$

has a horizontal vanishing line at E_∞, and one deduces that the translation of this $\frac{1}{11}$-vanishing line passing through $(t-s, s) = (21, 3)$ (the bidegree of $h_{1,3}^3$) serves as a vanishing line for ${}^{ass}E_2^{*,*}(A_2)$. □

[6] Although we use similar notation for generators in $H^{*,*,*}(\mathcal{C}_{alg}(Z))$, we warn the reader that in this context we are using Adams-Novikov naming conventions, so that the May spectral sequence generator $h_{i,j}$ corresponds to the generator $h_{i,j-1} \in H^{*,*,*}(\mathcal{C}_{alg}(Z))$ for $i \geq 2$ and $j \in \{1, 2\}$.

Proposition 6.2.4. *The evil complex $V^{n,t}(Z)$ satisfies*

$$H^{n,t}(V^{*,*}) = 0$$

for

$$n > \frac{(t-n)+12}{11}.$$

Proof. We explain the relationship between $H^{*,*}(V)$ and ${}^{ass}E_2^{*,*}(A_2)$ by constructing a spectral sequence which relates them. We first note that because $H^*(A_2) \cong A(2)$, we have

$${}^{\mathrm{tmf}}_{alg}E_1^{n,s,t}(A_2) = 0$$

for $s > 0$. Therefore, the only possible non-trivial differentials in the tmf-MSS are d_1 differentials, and

$${}^{\mathrm{tmf}}_{alg}E_2^{n,0,t}(A_2) \cong {}^{ass}E_2^{n,t}(A_2).$$

The short exact sequence of A_*-comodules

$$0 \to H_*Z \to H_*A_2 \to H_*\Sigma^7 Z \to 0$$

induces a long exact sequence

$$0 \to {}^{\mathrm{tmf}}_{alg}E_1^{n,0,t}(Z) \to {}^{\mathrm{tmf}}_{alg}E_1^{n,0,t}(A_2) \to {}^{\mathrm{tmf}}_{alg}E_1^{n,0,t-7}(Z) \xrightarrow{v_2} {}^{\mathrm{tmf}}_{alg}E_1^{n,1,t}(Z) \to \cdots$$

We therefore deduce that there is a short exact sequence

$$0 \to {}^{\mathrm{tmf}}_{alg}E_1^{n,0,t}(Z) \to {}^{\mathrm{tmf}}E_1^{n,0,t}(A_2) \to V^{n,t-7}(Z) \to 0.$$

This allows us to consider the decreasing filtration of cochain complexes, with associated filtration quotients:

$$\begin{tikzcd}
{}^{\mathrm{tmf}}_{alg}E_1^{n,0,t}(A_2) \arrow[d, two heads] & {}^{\mathrm{tmf}}_{alg}E_1^{n,0,t}(Z) \arrow[l, hook'] \arrow[d, two heads] & V^{n,t}(Z) \arrow[l, hook'] \arrow[d, equal] & 0 \arrow[l] \\
V^{n,t-7}(Z) & \mathcal{C}_{alg}^{n,0,t}(Z) & V^{n,t}(Z) &
\end{tikzcd}$$

Taking cohomology, we get a strange little spectral sequence which we will dub the *algebraic AKSS for A_2* as it more or less arises as a kind of mod v_2 version of the algebraic AKSS for Z. If we index it as follows:[7]

$$\begin{aligned}
{}^{akss}_{alg}E_{1+\epsilon}^{n-\epsilon,t}(A_2) &= H^{n,t-7}(V), \\
{}^{akss}_{alg}E_{1+\epsilon}^{n,t}(A_2) &= H^{n,0,t}(\mathcal{C}_{alg}), \\
{}^{akss}_{alg}E_{1+\epsilon}^{n+\epsilon,t}(A_2) &= H^{n,t}(V),
\end{aligned}$$

then the resulting spectral sequence takes the form

$${}^{akss}_{alg}E_{1+\epsilon}^{n+\alpha\epsilon,t}(A_2) \Rightarrow {}^{ass}E^{n,t}(A_2)$$

[7] With this indexing convention the map $Z \to A_2$ results in a map of spectral sequences ${}^{akss}_{alg}E_*^{n+\alpha\epsilon,s,t}(Z) \to {}^{akss}_{alg}E_*^{n+\alpha\epsilon,t}(A_2)$ (which one takes to be the zero map on terms with $s > 0$).

with differentials

$$\begin{aligned} d_{1+\epsilon} &: {}^{akss}_{alg}E_{1+\epsilon}^{n-\epsilon,t}(A_2) \to {}^{akss}_{alg}E_{1+\epsilon}^{n+1,t}(A_2) \\ d_{1+\epsilon} &: {}^{akss}_{alg}E_{1+\epsilon}^{n,t}(A_2) \to {}^{akss}_{alg}E_{1+\epsilon}^{n+1+\epsilon,t}(A_2) \\ d_{1+2\epsilon} &: {}^{akss}_{alg}E_{1+2\epsilon}^{n-\epsilon,t}(A_2) \to {}^{akss}_{alg}E_{1+2\epsilon}^{n+1+\epsilon,t}(A_2) \end{aligned}$$

and

$${}^{akss}_{alg}E_2^{n+\alpha\epsilon,t}(A_2) = {}^{akss}_{alg}E_\infty^{n+\alpha\epsilon,t}(A_2).$$

The result follows for dimensional reasons (by induction on $t-n$) using Lemma 6.2.3 and the fact that

$$H^{n,0,t}(\mathcal{C}_{alg}) = 0$$

for

$$n > \frac{t-n}{11}$$

(since the generator of $H^{*,*,*}(\mathcal{C}_{alg})$ with lowest slope is $\widetilde{h}_{2,1}$, with slope $\frac{n}{t-n} = \frac{1}{11}$). □

Proposition 6.2.5. *The map*

$${}^{ass}E_2^{s,t}(Z) \to v_2^{-1}\,{}^{ass}E_2^{s,t}(Z)$$

is an isomorphism for

$$s > \frac{(t-s)+12}{11}.$$

Proof. The result follows from considering the map of algebraic AKSS's

$${}^{akss}_{alg}E_*^{*,*,*}(Z) \to v_2^{-1}\,{}^{akss}_{alg}E_*^{*,*,*}(Z)$$

and using Proposition 6.2.1, Corollary 6.2.2, Proposition 6.2.4, and the observation that the map

$$H^{n,s,t}(\mathcal{C}_{alg}) \to v_2^{-1}H^{n,s,t}(\mathcal{C}_{alg})$$

is an isomorphism for

$$n+s > \frac{t-n-s}{11}.$$

□

Given a class $x \in {}^{ass}E_2(Z)$ it is therefore straightforward to determine from low dimensional computations if it is v_2-periodic Let k be chosen such that $v_2^k x$ lies in the range of Proposition 6.2.5. Then x is v_2-periodic if an only if $v_2^k x \neq 0$.

The following theorem, analogous to the dichotomy principle in [BBB+17], completely determines whether classes in ${}^{ass}E_2$ are good or evil. Note that because of Corollary 6.2.2 (which does not have an analog in the context studied in [BBB+17]), the proof of the dichotomy principle is much more straightforward in the present context.

Theorem 6.2.6 (Dichotomy Principle)**.** *Suppose that x is a non-trivial class in ${}^{ass}E_2^{s,t}(Z)$.*

(1) If x is v_2-torsion, it is evil.
(2) Every class in the range of Proposition 6.2.5 is good.

(3) *Suppose x is v_2-periodic, and suppose that k is taken large enough so that $v_2^k x$ lies in the range of Proposition 6.2.5. Suppose that $v_2^k x$ is detected in the AKSS by a class in ${}^{akss}_{alg}E_{1+\epsilon}^{n,*,*}$. Then x is good if and only if*

$$s \geq n.$$

Proof. We deduce (1) from Corollary 6.2.2. We deduce (2) from Proposition 6.2.4. For (3), suppose that x is v_2-periodic, detected by an evil class

$$\widetilde{x} \in {}^{akss}_{alg}E_{1+\epsilon}^{n'+\epsilon, s-n', t}$$

in the algebraic AKSS. Then we must have

$$s = n'.$$

Since $\widetilde{x}$ is v_2-torsion, we deduce that the v_2^k-multiplication must arise from a hidden extension in the AKSS, and therefore

$$s = n' < n.$$

Suppose however that x is detected by a good class

$$\widetilde{x} \in {}^{akss}_{alg}E_{1+\epsilon}^{n', s-n', t}.$$

Then we must have

$$s - n' \geq 0.$$

We deduce from the proof of Corollary 6.2.2 that $n' = n$, and therefore $s - n \geq 0$ and

$$s \geq n.$$

□

6.3. **The topological AKSS.** There is a topological analog of the AKSS, which refines the tmf-ASS just as the (algebraic) AKSS constructed in the beginning of this section refines the tmf-MSS.

Consider the short exact sequence

$$0 \to V^{n,*}(Z) \to {}^{\mathrm{tmf}}E_1^{n,*}(Z) \xrightarrow{g} \mathcal{C}^{n,*}(Z) \to 0. \tag{6.3.1}$$

Just as in the algebraic case, we will regard the evil subcomplex $V^{n,*}(Z)$ as being in filtration $n + \epsilon$. The result is a topological AKSS:

$$\{{}^{akss}E_{r+\beta\epsilon}^{n+\alpha\epsilon, t}\} \Rightarrow \pi_{t-n}(Z)$$

with differentials

$$\begin{aligned}
d_{r-\epsilon}^{akss} &: {}^{akss}E_{r-\epsilon}^{n+\epsilon,t} \to {}^{akss}E_{r-\epsilon}^{n+r,t}, \\
d_{r}^{akss} &: {}^{akss}E_{r}^{n+\alpha\epsilon,t} \to {}^{akss}E_{r}^{n+r+\alpha\epsilon,t}, \\
d_{r+\epsilon}^{akss} &: {}^{akss}E_{r+\epsilon}^{n,t} \to {}^{akss}E_{r+\epsilon}^{n+r+\epsilon,t}.
\end{aligned}$$

The E_1-term takes the form

$${}^{akss}E_1^{n+\alpha\epsilon,t} = \begin{cases} \mathcal{C}^{n,t}(Z), & \alpha = 0, \\ V^{n,t}(Z), & \alpha = 1. \end{cases}$$

The d_1-differential

$$d_1^{akss} : {}^{akss}E_1^{n+\alpha\epsilon,t} \to {}^{akss}E_1^{n+1+\alpha\epsilon,t}$$

is given by

$$d_1 = \begin{cases} d_1^{good}, & \alpha = 0, \\ d_1^{evil}, & \alpha = 1. \end{cases}$$

We therefore have

$$^{akss}E_{1+\epsilon}^{n+\alpha\epsilon,t} = \begin{cases} H^{n,t}(\mathcal{C}), & \alpha = 0, \\ H^{n,t}(V), & \alpha = 1. \end{cases}$$

The only nonzero $d_{1+\epsilon}$-differentials are of the form

$$H^{n,t}(\mathcal{C}) = {}^{akss}E_{1+\epsilon}^{n,t} \xrightarrow{d_{1+\epsilon}} {}^{akss}E_{1+\epsilon}^{n+1+\epsilon,t} = H^{n+1,t}(V),$$

for which we have

$$d_{1+\epsilon} = \partial$$

where ∂ is the connecting homomorphism of (3.3.2). It turns out all of these differentials can be derived from the algebraic AKSS.

Lemma 6.3.2. *For $n = 0$, the differentials*

$$d_{1+\epsilon} : {}^{akss}E_{1+\epsilon}^{n,t} \xrightarrow{d_{1+\epsilon}} {}^{akss}E_{1+\epsilon}^{n+1+\epsilon,t}$$

are trivial. For $n \geq 1$, they are determined by the following commutative diagram:

$$\begin{tikzcd} {}^{akss}E_{1+\epsilon}^{n,t} \arrow[r, "d_{1+\epsilon}"] \arrow[d] & {}^{akss}E_{1+\epsilon}^{n+1+\epsilon,t} \arrow[d, equal] \\ {}^{akss}_{alg}E_{1+\epsilon}^{n,0,t} \arrow[r, "d_{1+\epsilon}^{alg}"'] & {}^{akss}_{alg}E_{1+\epsilon}^{n+1+\epsilon,0,t} \end{tikzcd}$$

Proof. Topologically, $d_{1+\epsilon}$ derives from applying π_* to the composite

$$C^n \to \mathrm{tmf}^{\wedge n+1} \wedge Z \to \mathrm{tmf}^{\wedge n+2} \wedge Z \to HV^{n+1}. \tag{6.3.3}$$

The first statement follows from the fact that the only elements in $H^{n,*}(\mathcal{C})$ for $n = 0$ are powers of v_2. The second statement follows from the fact that $d_{1+\epsilon}^{alg}$ is the induced map of Adams $E_2^{0,*}$-terms coming from the composite (6.3.3):

$$\mathcal{C}_{alg}^{n,0,*} = {}^{ass}E^{0,*}(C^n) \to {}^{ass}E^{0,*}\left(HV^{n+1}\right) = V^{n+1,*}. \qquad \square$$

The E_2-term of the tmf-ASS is deduced from the short exact sequence

$$0 \to {}^{akss}E_2^{n+\epsilon,t} \to {}^{\mathrm{tmf}}E_2^{n,t} \to {}^{akss}E_2^{n,t} \to 0.$$

The differentials in the topological AKSS determine and are determined by the differentials in the tmf-ASS, with lengths dictated by whether the sources and targets of the tmf-ASS differentials are good or evil. Unlike the algebraic case, in the topological case there are no dimensional restrictions: in principle good or evil classes can each kill either good or evil classes (but not via d_1-differentials since $V^{*,*}(Z)$ is a subcomplex of the E_1-term). Furthermore, there is no dichotomy principle in the topological AKSS.

7. Stem by stem computations

In this section, we apply the agathokakological techniques of the previous section to do low dimensional computations of $\pi_* Z$. Furthermore, we settle the ambiguity left in [BE16b] regarding the differentials in the $K(2)$-local Adams Novikov spectral sequence (Theorem 7.5.1).

7.1. **The algebraic AKSS.**

In this section, we use the algebraic AKSS

$$\{{}^{akss}_{alg}E^{n+\alpha\epsilon,s,t}_{r+\beta\epsilon}(Z)\} \Rightarrow {}^{ass}E_2^{n+s,t}(Z)$$

to identify $H^*(V(Z))$ in the range relevant for computing $\pi_* Z$ in degrees $* \leq 39$.

More specifically, we do these computations for a specific choice of Z and v_2-self map. It is shown in [BE16a][§2] that for any $Z \in \widetilde{\mathcal{Z}}$ and v_2^1-self map $f\colon \Sigma^6 Z \to Z$, there is a cofiber sequence

$$\Sigma^6 Z \xrightarrow{f} Z \longrightarrow C(f) \longrightarrow \Sigma^7 Z \tag{7.1.1}$$

where $C(f)$ is a spectrum with the property that $H^* C(f)$ is isomorphic to $A(2)$ as an $A(2)$-module. The different choices of $Z \in \widetilde{\mathcal{Z}}$ and v_2^1-self maps give rise to different A-module structures on $A(2)$.

For the rest of the section we work with those $Z \in \widetilde{\mathcal{Z}}$ whose A-module structure is the one given in [BE16a, Appendix 1]. We will denote this A-module $B(2)$.[8] From [BE16a, Remark 5.4], we learn that there are four different homotopy types of *finite* spectra realizing $B(2)$. Of course all of them support a v_2^1-self map by [BE16a, Main Theorem 1]. In [BE16a], the authors define $B(2)$ as

$$B(2) := A(2) \otimes_{E(Q_2)} \mathbb{F}_2,$$

where $A(2)$ is the A-module of [Rot77, p.30]. See also Appendix A. It follows that the cofiber of any v_2^1-self map of our chosen Z is a realization of the module $A(2)$. It turns out there is a unique homotopy type of spectra realizing $A(2)$ ($\mathrm{Ext}_A^{s,s+1}(A(2), A(2)) = 0$ for $s \geq 2$). Therefore, different choices of a v_2^1-self map on our chosen Z will not affect the calculations that follow. For this choice, we let

$$A_2 := C(f).$$

In this section, we also define

$$\mathrm{Ext}_A^{s,t}(Z) := \mathrm{Ext}_A^{s,t}(H^*(Z), \mathbb{F}_2), \qquad \mathrm{Ext}_A^{s,t}(A_2) := \mathrm{Ext}_A^{s,t}(H^*(A_2), \mathbb{F}_2).$$

Both $\mathrm{Ext}_A^{*,*}(Z)$ and $\mathrm{Ext}_A^{*,*}(A_2)$ can be computed using Bruner's program [Bru93]. The results are depicted in Figure 7.1 and Figure 7.2 in Adams grading $(x, y) = (t - s, s)$.

---

[8] In [BE16a], $A(2)$ is denoted by A_2 and $B(2)$ by B_2.

7.2. **v_2-multiplication in $\mathrm{Ext}_A(Z)$.** To proceed with our computations, we will need to determine which classes in $\mathrm{Ext}_A^{*,*}(Z)$ are detected by evil classes, and which are detected by good classes. This will be done using the dichotomy principle (Theorem 6.2.6), and so we need to identify the v_2-periodic classes in $\mathrm{Ext}_A^{*,*}(Z)$. To do this, we proceed as follows.

Note that there is a long exact sequence
(7.2.1)

$$\dots \longrightarrow \mathrm{Ext}_A^{s,t}(Z) \longrightarrow \mathrm{Ext}_A^{s,t}(A_2) \longrightarrow \mathrm{Ext}_A^{s,t}(\Sigma^7 Z) \xrightarrow{\delta} \mathrm{Ext}_A^{s+1,t}(Z) \longrightarrow \dots$$

where the connecting homomorphism δ corresponds to multiplication by v_2,

$$\delta = v_2\colon \mathrm{Ext}_A^{s,t}(\Sigma^7 Z) \cong \mathrm{Ext}_A^{s,t-7}(Z) \to \mathrm{Ext}_A^{s+1,t}(Z).$$

The v_2-multiplications in $\mathrm{Ext}_A^{*,*}(Z)$ are indicated by dotted lines of slope $(6,1)$ in Figures 7.1 and 7.2. The indicated multiplications are completely determined by the long exact sequence (7.2.1). In Example 7.2.2, we give a sample proof deducing the existence of a v_2-multiplication from the long exact sequence. The proofs for the other v_2-multiplications indicated in Figures 7.1 and 7.2 are also straightforward, though the arguments involving classes in stems $* \geq 40$ become more tedious due to the growing dimension of $\mathrm{Ext}_A^{*,*}(A_2)$ and of $\mathrm{Ext}_A^{*,*}(Z)$. The v_2-multiplication data in Figures 7.1 and 7.2 is complete in stems $x \leq 39$. In stems $40 \leq x \leq 60$, we only draw those multiplications which are necessary to apply part (3) of Theorem 6.2.6 to do computations up to $* = 39$.

Example 7.2.2. If x is the non-zero class in $(t-s,s) = (15,1)$ of $\mathrm{Ext}_A^{*,*}(Z)$, then $v_2 x \neq 0$. Indeed, in degree $(t-s,s) = (21,2)$ (the target of v_2-multiplication on x), $\mathrm{Ext}_A^{*,*}(A_2)$ is one dimensional over $\mathbb{F}_2$. However, there are two possible contributions to $\mathrm{Ext}_A^{*,*}(A_2)$ in this degree from the long exact sequence (7.2.1). (See Figure 7.3 and its caption.) There is a class $\Sigma^7 y$ of $\mathrm{Ext}_A^{*,*}(\Sigma^7 Z)$, labeled $\bullet 1$ of Figure 7.3, where y is the class labeled $1\bullet$ in degree $(14,2)$ of Figure 7.3. There is also a class z of $\mathrm{Ext}_A^{*,*}(Z)$, labeled $\bullet 6$ in Figure 7.3. Since $v_2 y = 0$ for degree reasons, $\Sigma^7 y$ is in the kernel of the connecting homomorphism δ. Therefore, the non-zero element of $\mathrm{Ext}_A^{*,*}(A_2)$ corresponds to the class $\Sigma^7 y$. For degree reasons, $\delta(z) = 0$, and so there must be a class w of degree $(22,1)$ in $\mathrm{Ext}_A^{*,*}(\Sigma^7 Z)$ such that $\delta(w) = z$. The only possibility is the class labeled by $\bullet 4$ of Figure 7.3. The class x corresponds to $4\bullet$ in Figure 7.3, and so $w = \Sigma^7 x$. It follows that $v_2 x = z$.

7.3. **The differentials in the algebraic AKSS.** We turn to the computation of the algebraic AKSS. From Remark 5.4.1, we have that

$$H^{*,*,*}(\mathcal{C}_{alg}) \cong \mathbb{F}_2[v_2, \widetilde{h}_{2,1}, h_{3,0}, h_{3,1}, h_{4,0}, h_{4,1}, \dots]. \tag{7.3.1}$$

We use the dichotomy principle to determine which classes of $\mathrm{Ext}_A(Z)$ are good and which are evil. With (7.3.1) and the results of the previous section on v_2-multiplications, this is straightforward and result of this analysis is depicted in Figure 7.4.

Having determined which classes in $\mathrm{Ext}_A^{*,*}(Z)$ are detected by good and evil, we can now deduce $H^{*,*}(V)$ from the algebraic AKSS. We name the evil classes in the

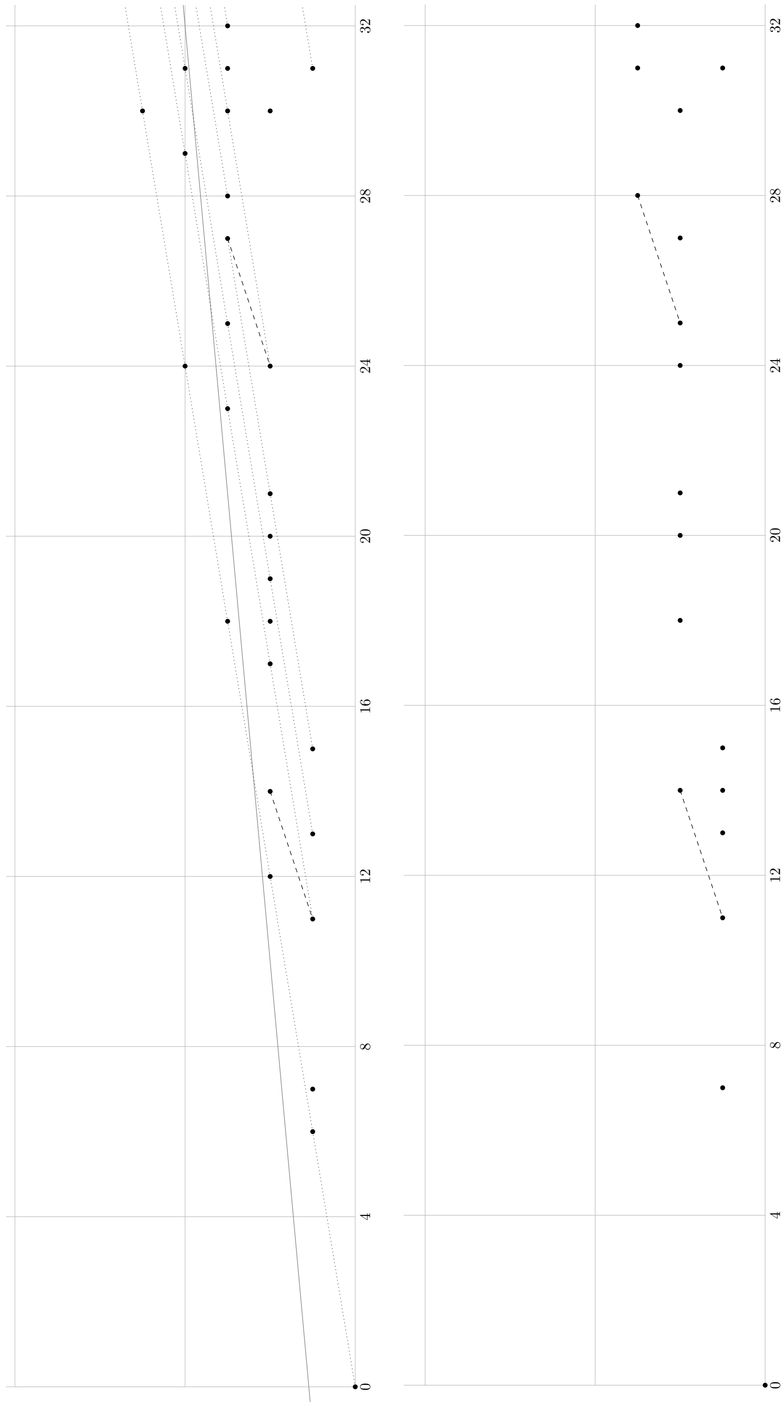

FIGURE 7.1. $\mathrm{Ext}_A^{s,t}(Z)$ (left) and $\mathrm{Ext}_A^{s,t}(A_2)$ (right) drawn in Adams coordinates $(x, y) = (t - s, s)$ in degrees $x \leq 32$. The dotted lines of slope $(6, 1)$ denote v_2-multiplication. The solid lines of slope $(1, 1)$ denote h_1 (i.e. η) multiplications and those of slope $(3, 1)$ denote h_2 (i.e. ν) multiplications. The gray line of slope $1/11$ is the line of Proposition 6.2.5.

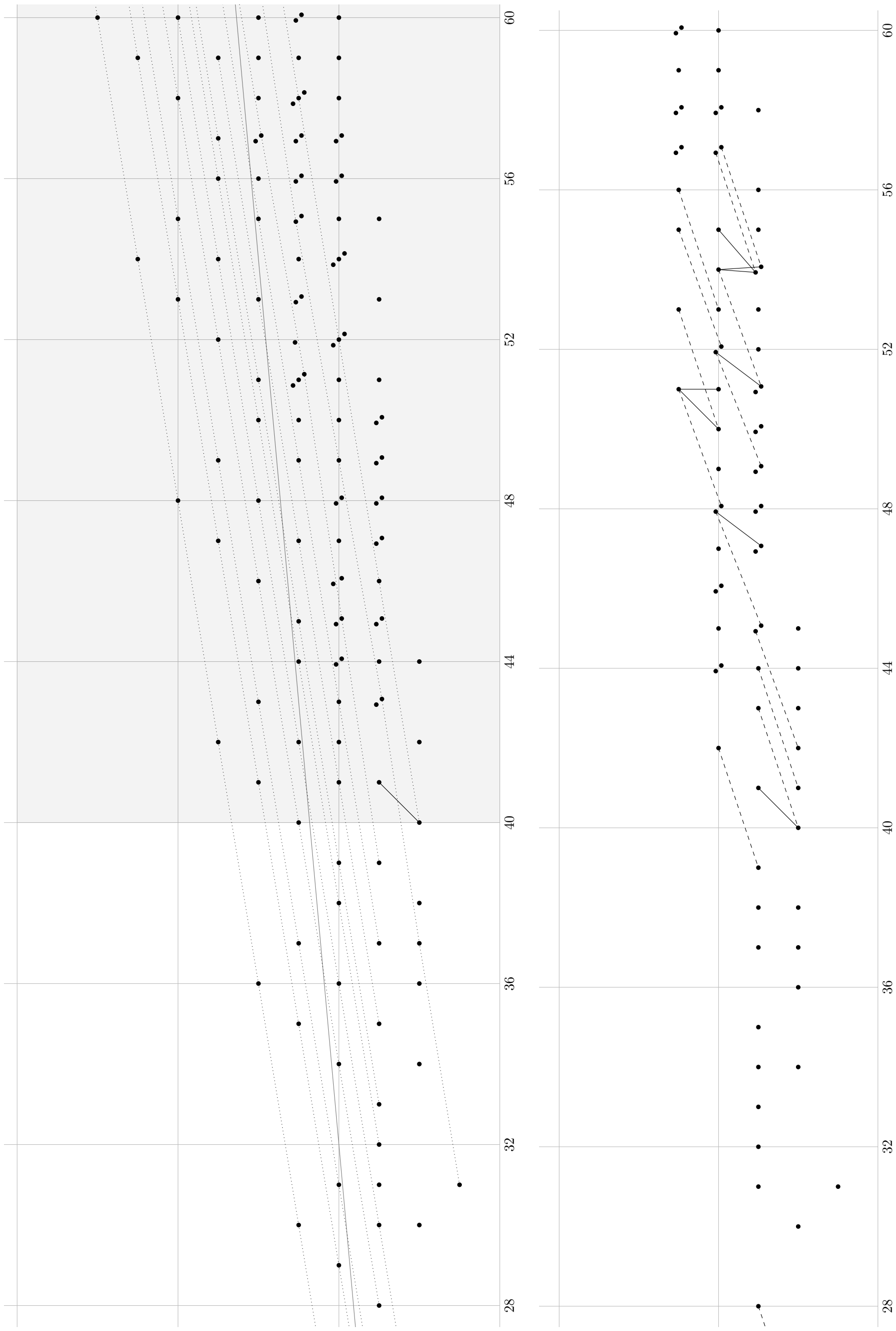

FIGURE 7.2. $\mathrm{Ext}_A^{s,t}(Z)$ (left) and $\mathrm{Ext}_A^{s,t}(A_2)$ (right) drawn in Adams coordinates $(x, y) = (t - s, s)$ in degrees $28 \leq x \leq 60$. In $\mathrm{Ext}_A^{s,t}(Z)$, not all v_2-multiplications are drawn in the shaded area, but we have included those needed for our computation. The gray line of slope $1/11$ is the line of Proposition 6.2.5.

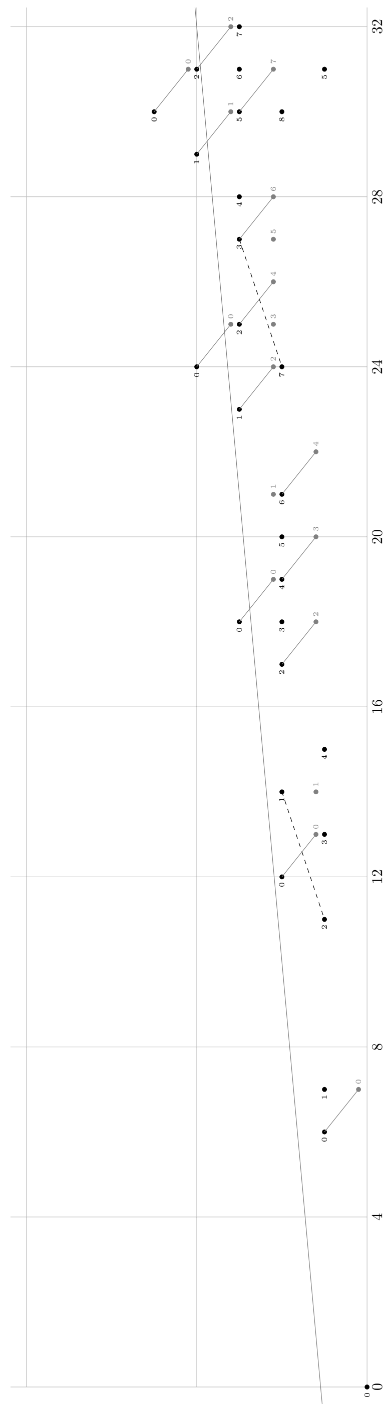

FIGURE 7.3. The connecting homomorphism $\mathrm{Ext}_A^{s,t}(\Sigma^7 Z) \to \mathrm{Ext}_A^{s+1,t}(Z)$. The gray classes are elements of $\mathrm{Ext}_A^{s,t}(\Sigma^7 Z)$, the black classes are elements of $\mathrm{Ext}^{s+1,t}(Z)$. The gray lines of slope $(-1,1)$ give the connecting homomorphism, which in turn corresponds to v_2-multiplication. The gray line of slope $1/11$ is the line of Proposition 6.2.5.

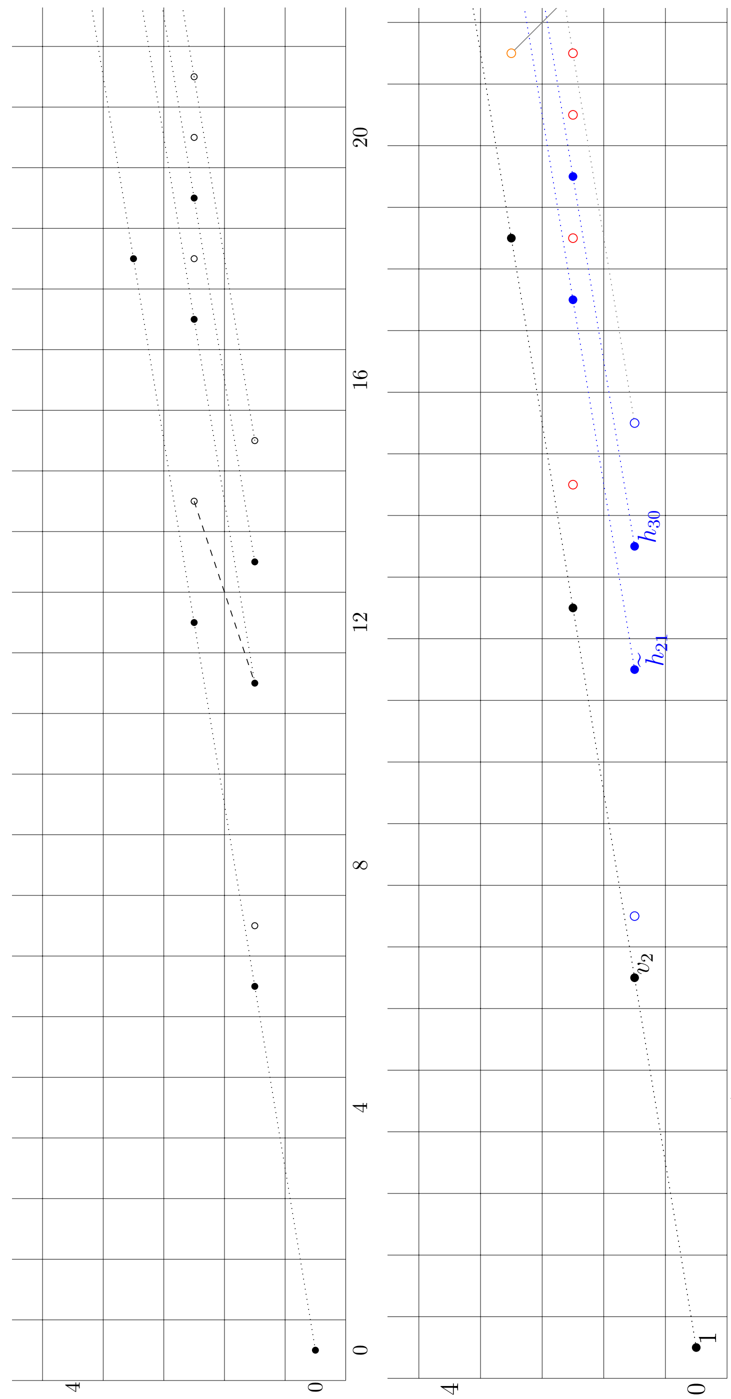

FIGURE 7.4. The left chart is the E_2-term of the ASS for Z in stems $0 \leq t - s \leq 21$. Classes detected by good are denoted by $\bullet$ and classes detected by evil by $\circ$. The right chart is the algebraic AKSS for Z, starting at the $E_{1+\epsilon}$-page.

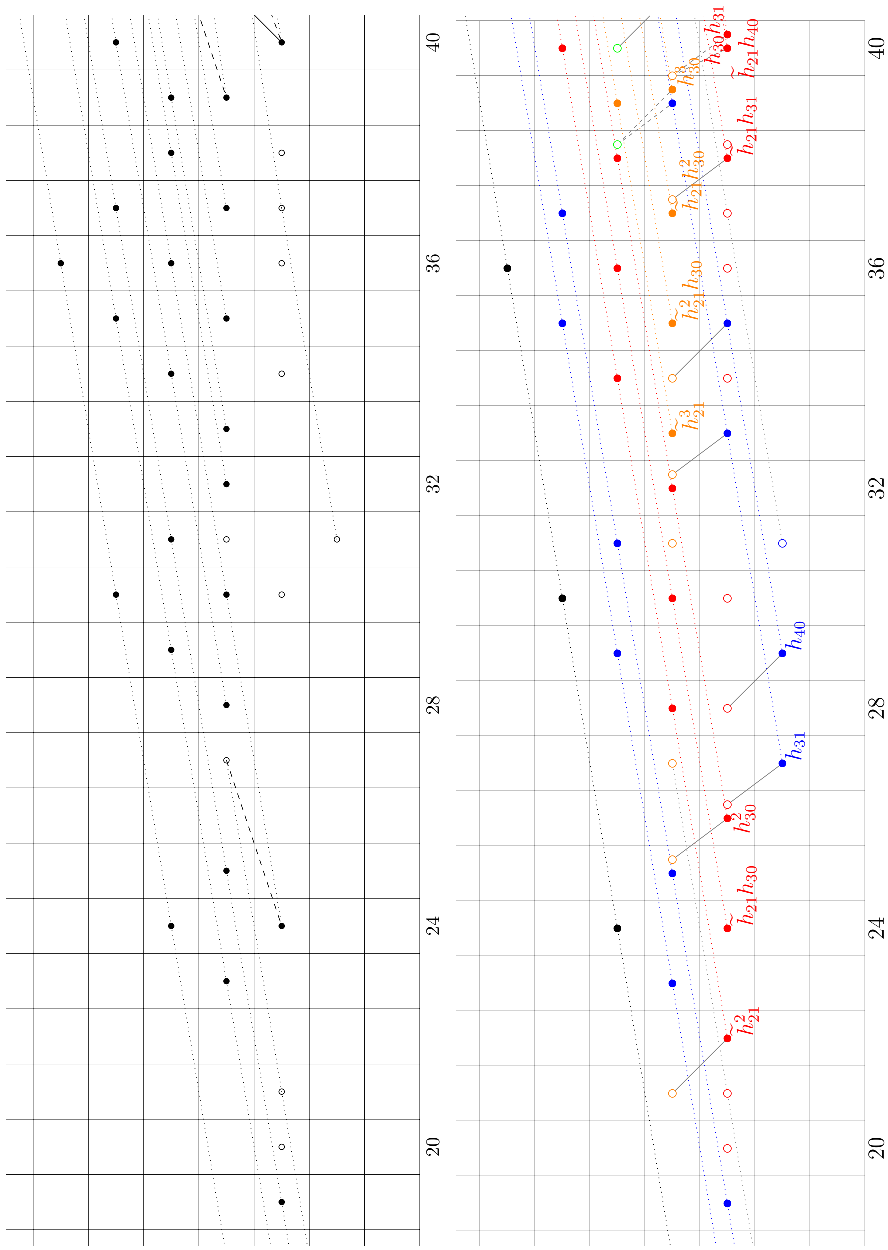

FIGURE 7.5. The left chart is the E_2-term of the ASS for Z in stems $19 \leq t-s \leq 40$. Classes detected by good are denoted by $\bullet$ and classes detected by evil by $\circ$. The right chart is the algebraic AKSS for Z, starting at the $E_{1+\epsilon}$-page.

n	color
0	black
1	blue
2	red
3	orange
4	green

TABLE 1. The tmf-filtration.

algebraic AKSS (Figure 7.4) by

$$(x, y : n)^{ev},$$

where $(x, y) = (t - (s + n), s + n)$ is the Adams coordinate and n is the tmf-filtration. These classes are denoted by open circles in Figure 7.4. The good classes are denoted by solid circles. For example, the class in degree $(x, y) = (7, 1)$ in $\mathrm{Ext}_A(Z)$ is detected by evil and denoted by $(7, 1 : 1)^{ev}$ in the algebraic AKSS.

In stems $0 \leq x \leq 39$, the following evil classes exist for degree reasons. More precisely, these evil classes detect a class in $\mathrm{Ext}_A(Z)$ in a degree which contains no non-zero element of $H^*(\mathcal{C}_{alg})$:

$(7, 1 : 1)^{ev}$	$(14, 2 : 2)^{ev}$	$(27, 3 : 3)^{ev}$
$(15, 1 : 1)^{ev}$	$(18, 2 : 2)^{ev}$	$(31, 3 : 3)^{ev}$
$(31, 1 : 1)^{ev}$	$(20, 2 : 2)^{ev}$	
	$(21, 2 : 2)^{ev}$	
	$(30, 2 : 2)^{ev}$	
	$(34, 2 : 2)^{ev}$	
	$(36, 2 : 2)^{ev}$	
	$(37, 2 : 2)^{ev}$	
	$(38, 2 : 2)^{ev}$	

The following evil classes exist because of the following differentials

$$d_{1+\epsilon}(\widetilde{h}_{2,1}^2) = (21, 3 : 3)^{ev}$$
$$d_{1+\epsilon}(h_{3,0}^2) = (25, 3 : 3)^{ev}$$
$$d_{1+\epsilon}(h_{3,1}) = (26, 2 : 2)^{ev}$$
$$d_{1+\epsilon}(h_{4,0}) = (28, 2 : 2)^{ev}$$
$$d_{2+\epsilon}(v_2 h_{3,1}) = (32, 3 : 3)^{ev}$$
$$d_{2+\epsilon}(v_2 h_{4,0}) = (34, 3 : 3)^{ev}$$
$$d_{1+\epsilon}(\widetilde{h}_{2,1} h_{3,1}) = (37, 3 : 3)^{ev}$$
$$d_{3+\epsilon}(v_2^2 h_{4,0}) = (40, 4 : 4)^{ev}.$$

Examples of how we deduce these differentials is given in Example 7.3.2.

Example 7.3.2. In degree $(t - s, s) = (26, 2)$, $\mathrm{Ext}_A(Z)$ is trivial. Therefore, $h_{3,0}^2$ cannot survive the spectral sequence so must support a differential. Since the class in $(25,3)$ of $\mathrm{Ext}_A(Z)$ is detected by a good class, the only good class in that bidegree ($v_2^2 h_{3,0}$) cannot be hit by a differential. So the target of the differential on $h_{3,0}^2$ must be evil, and we obtain the differential

$$d_{1+\epsilon}(h_{3,0}^2) = (25,3:3)^{ev}.$$

The only non-trivial class in degree $(38,2)$ of $\mathrm{Ext}_A(Z)$ is detected by evil. Therefore $\widetilde{h}_{2,1}h_{3,1}$ must support a non-trivial differential. A similar analysis as before gives the differential

$$d_{1+\epsilon}(\widetilde{h}_{2,1}h_{3,1}) = (37,3:3)^{ev}.$$

Furthermore,

$$d_{1+\epsilon}(h_{3,0}h_{3,1}) = \alpha_1(39,3:3)^{ev} \quad \text{and} \quad d_{1+\epsilon}(\widetilde{h}_{2,1}h_{4,0}) = \alpha_2(39,3:3)^{ev} \tag{7.3.3}$$

where at least one of the coefficients α_i is non-zero. Similarly, at least one of the following $d_{2+\epsilon}$-differentials must occur

$$d_{3+\epsilon}(v_2^2 h_{3,1}) = (38,4:4)^{ev} \qquad \text{or} \qquad d_{1+\epsilon}(h_{3,0}^3) = (38,4:4)^{ev}$$

These ambiguities will be mostly settled in the next section.

7.4. **The topological AKSS and the computation of the tmf-based ASS for Z.** Now, we turn to our analysis of the spectral sequence

$${}^{\mathrm{tmf}}E_1^{n,t} = \pi_t(\mathrm{tmf}^{\wedge n+1} \wedge Z) \Longrightarrow \pi_{t-n}(Z)$$

and low-dimensional computations of $\pi_* Z$. Our analysis of the algebraic AKSS has allowed us to identify $H^{*,*}(V)$, together with the boundary homomorphism

$$H^{*,*,*}(\mathcal{C}_{alg}) \xrightarrow{\partial_{alg}} H^{*,*}(V)$$

in the form of $d_{1+\epsilon}$ differentials in the algebraic AKSS. Theorem 5.4.3 gives the E_1-term of the May-Ravenel SS

$${}^{MR}E_1(\widetilde{\sigma}(2)) \Rightarrow H^{*,*}(\mathcal{C}). \tag{7.4.1}$$

It does not exclude the possibility of differentials, but there are no possibilities of differentials in the range of interest.

We recall from Sections 3 and 6 that, in the tmf-based ASS, the following d_1 differentials have already been determined by our previous computation.

- An evil class cannot kill a good class via a d_1-differential since $V^{*,*}(Z)$ is a subcomplex of ${}^{\mathrm{tmf}}E_1^{*,*}(Z)$.
- The d_1-differentials between evil classes are completely determined by those in the algebraic AKSS since $V^{*,*}(Z) \cong V_{alg}^{*,0,*}(Z)$.
- The d_1-differentials from good classes to evil classes are determined by the differentials in the algebraic AKSS. This is Lemma 6.3.2.

In Figure 7.6, we draw ${}^{MR}E_1(\widetilde{\sigma}(2))$ in the range $0 \le t-n \le 40$, together with the information about $H^{*,*}(V)$ and differentials obtained from the algebraic AKSS.

We use the map of spectral sequences from tmf-based ASS to the classical ASS to ascertain that, in the range $t-s \le 39$, there are no additional differentials.

Proposition 7.4.2. *There are no non-trivial differentials in the classical ASS for Z with source in stem $t-s \le 39$.*

Proof. In the computations of $\pi_* Z$ for $0 \le * \le 39$, the possible differentials have source in stems

$$t-s = 30, 31, 36, 37, 38, 40.$$

In stems $t-s < 40$, the potential sources for differentials are the image of evil classes which are permanent cycles in the tmf-based ASS. Indeed, for degree reasons, these classes are permanent cycles provided that they are d_1-cycles. Since all d_1-differentials on evil classes have been recorded in Figure 7.6 and all of the potential sources are d_1-cycles, the claim follows. □

Remark 7.4.3. There is a potential d_2-differential in stem $t-s = 40$ in the classical ASS for Z. In fact, this problem is tied to the ambiguity in (7.3.3), as we will see in the proof of the next proposition, where we will establish that such a non-trivial d_2 differential must occur in the ASS for Z.

Proposition 7.4.4. *The only non-trivial differential d_r for $r > 1$ in the* tmf*-based ASS with source in the range $t-n \le 40$ is*

$$d_2(v_2 h_{3,1}) = (32, 3:3)^{ev}.$$

Proof. Combining degree arguments with v_2-linearity, the only two possibilities are

$$d_2(v_2 h_{3,1}) = (32, 3:3)^{ev},$$
$$d_3(v_2^2 h_{3,1}) = (38, 4:4)^{ev}.$$

By Proposition 7.4.2, the classical ASS for Z collapses in this range. Therefore, $\pi_{32} Z$ and $\pi_{33} Z$ have order 2. For this to be the case, we must have $d_2(v_2 h_{3,1}) = (32, 3:3)^{ev}$ in the tmf-based ASS.

Since $\widetilde{h}_{2,1} h_{4,0}$ is not an element in $H^{*,*}(\mathcal{C})$, if

$$d_{1+\epsilon}(h_{3,0} h_{3,1}) = 0,$$

then it follows from the tmf-based ASS that we must have

$$d_3(v_2^2 h_{3,1}) = (39, 3:3)^{ev}$$

and $\pi_{39} Z$ has order 4. If, however, the correct differential is

$$d_{1+\epsilon}(h_{3,0} h_{3,1}) = (39, 3:3)^{ev},$$

then it follows from the tmf-based ASS that $\pi_{39} Z$ has order 2.

From the structure of the tmf-ASS we deduce that the map

$$v_2 : \pi_{33}(Z) \xrightarrow{v_2} \pi_{39}(Z)$$

is zero. It is immediate from Figure 7.2 that the ASS for A_2 collapses in degree 39 to give

$$\pi_{39}(A_2) = \mathbb{Z}/2.$$

It follows from the long exact sequence associated to the cofiber sequence

$$\Sigma^6 Z \xrightarrow{v_2} Z \to A_2$$

that we must have

$$\pi_{39}(Z) = \mathbb{Z}/2.$$

We therefore conclude that

$$d_{1+\epsilon}(h_{3,0}h_{3,1}) = (39, 3:3)^{ev}$$

and

$$d_3(v_2^2 h_{3,1}) = 0.$$

□

It follows from Proposition 7.4.2 and Proposition 7.4.4 that Figure 7.6 is complete.

7.5. **The $K(2)$-localization of Z.** We end this section with one of the main goals of this paper, which is to determine the homotopy groups of $\pi_* Z_{K(2)}$.

Theorem 7.5.1. *The $K(2)$-local Adams Novikov spectral sequence for Z collapses at the E_2-term.*

Proof. This spectral sequence is isomorphic (E_2 onwards) to the v_2-localized tmf-ASS

$$v_2^{-1}\,{}^{\mathrm{tmf}}E_1^{n,t}(Z) \Longrightarrow \pi_{t-n} Z_{K(2)}.$$

The E_2-term is given by inverting v_2 in Theorem 5.3.2, and so is isomorphic to

$$\mathbb{F}_2[v_2^{\pm 1}] \otimes E(h_{3,0}, \widetilde{h}_{2,1}, h_{3,1}, \widetilde{h}_{4,1})$$

(see Figure 7.7). All differentials are v_2-linear since $Z_{K(2)}$ has a v_2^1-self map. Furthermore, there is a horizontal vanishing line at E_2. Indeed, $E_2^{n,t} = 0$ for $n \geq 5$. The class labeled by 1 is the image of $\pi_0 S^0 \to \pi_0 Z_{K(2)}$ so is a permanent cycle. For degree reasons, the only possible non-trivial differentials are d_3's with sources $v_2^k h_{31}$. However, since $d_3(v_2^2 h_{3,1})$ in the tmf-based ASS is zero, $v_2^2 h_{3,1}$ maps to a d_3-cycle in $v_2^{-1}\,{}^{\mathrm{tmf}}E_1^{n,t}$. □

Next, we solve all but one exotic extension:

Theorem 7.5.2. *For $k \not\equiv 3 \mod 6$, the groups $\pi_k Z_{K(2)}$ are annihilated by multiplication by 2.*

Proof. The class detected by $\widetilde{h}_{2,1}$ in $\pi_{11} Z$ and $h_{3,0}$ in $\pi_{13} Z$ have order 2 since there is no room in the tmf-based ASS for exotic extensions in these degrees. Therefore, their images in $\pi_* Z_{K(2)}$ have order 2, and so do all their multiples. The class detected by $v_2^{-10}\widetilde{h}_{4,1}$ is in the image of the bottom cell, $S^0_{K(2)} \to Z_{K(2)}$. Indeed, it is the image of the element $\zeta_2 \in \pi_{-1} L_{K(2)} S^0$ discussed in [DH04, Proposition 2.2.1].[9] So, any multiple of $v_2^{-10}\widetilde{h}_{4,1}$ has order 2. □

[9] Our notation differs from [Rav77, (3.4) Theorem]. In this reference, our class $v_2^{-10}\widetilde{h}_{4,1}$ is closely related to ρ_2 and Ravenel's ζ_2 is closely related to $v_2^{-2}\widetilde{h}_{2,1}$.

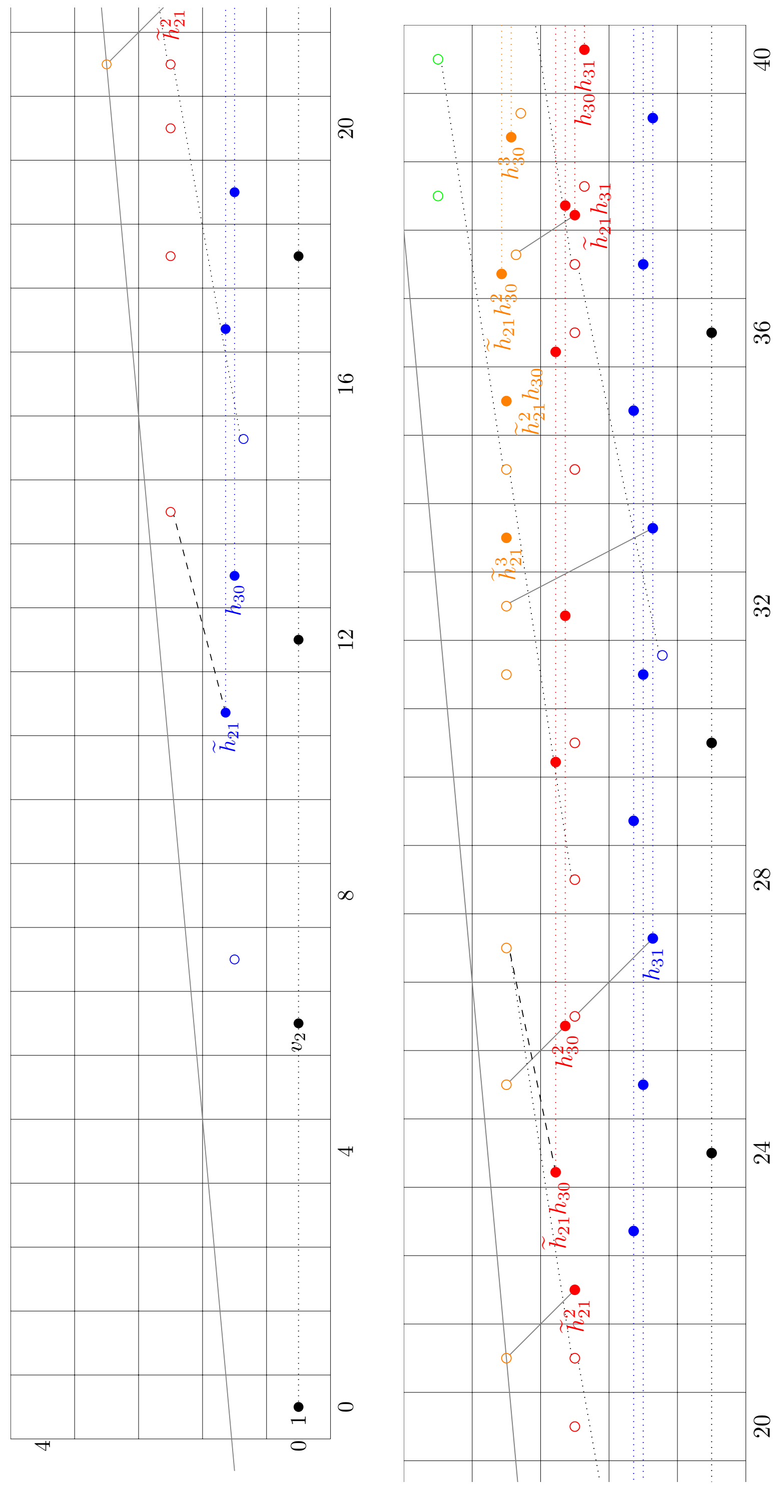

FIGURE 7.6. The topological AKSS computing $\pi_{t-n}(Z)$ drawn in grading $(x, y) = (t - n, n)$, starting at the $E_{1+\epsilon}$-page. Gray lines are differentials. They are dashed, if our method is inconclusive. Dotted lines are known v_2-multiplications. Dashed line are known ν-multiplications. The gray line of slope 1/11 is the line of Proposition 6.2.5.

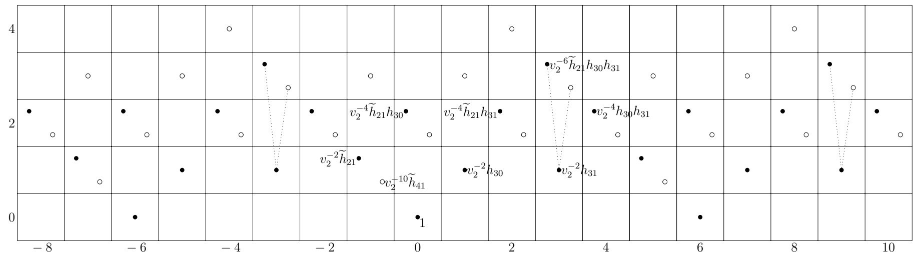

FIGURE 7.7. The E_∞-page of the $K(2)$-local E_2-based Adams Novikov spectral sequence for Z. The only possible non-trivial multiplication by 2 extensions are dotted. Classes denoted by $\circ$ are multiples of $\zeta_2 \in \pi_{-1}S^0_{K(2)}$.

Remark 7.5.3. In [BE16b], the authors study the $K(2)$-local E_2-based Adams Novikov spectral sequence for Z, where $K(2)$ is the Morava K-theory whose formal group law is the Honda formal group law. Since the homotopy type of $Z_{K(2)}$ is independent of the choice of $K(2)$, Theorem 7.5.1 and Theorem 7.5.2 settle Conjecture 1 of [BE16b] for our particular choice of $Z \in \widetilde{\mathcal{Z}}$, except for the group structure of $\pi_{3+6n}Z_{K(2)}$.

8. Discussion of the telescope conjecture for Z.

While the telescope conjecture was initially proposed by Ravenel [Rav84], Ravenel was also the first to propose that it should be false for chromatic levels ≥ 2 [Rav92b]. The method of disproof proposed in [Rav92b] (the *parameterized Adams spectral sequence*) turned out to not be sufficient to provide a counterexample to the telescope conjecture, but it laid out the blueprint for what could go wrong.

A more detailed account of this story is laid out by Mahowald-Ravenel-Shick [MRS01], who studied a family of Thom spectra $y(n)$ (defined for all primes p and all $n \geq 1$) and some conjectures about their localized Adams spectral sequences, which, if true, would provide counterexamples to the telescope conjecture for all primes p and all $n \geq 2$. These conjectures lay the groundwork for a concrete counter-conjecture for the homotopy of the telescopes proposed by Ravenel in [Rav95], which we shall call the *parabola conjecture.*

In this section we outline the analog of this conjectural story for Z, and explain how the structure of the tmf-ASS for Z described in this paper is consistent with the parabola conjecture. Specifically, let $\widehat{Z}$ denote the telescope of the v_2-self map on Z. The telescope conjecture predicts that the map

$$\widehat{Z} \to Z_{K(2)} \tag{8.0.1}$$

is an equivalence. In Theorem 7.5.1, we have already verified (up to a potential additive extension) that

$$\pi_* Z_{K(2)} \cong \mathbb{F}_2[v_2^{\pm 1}] \otimes E(\widetilde{h}_{2,1}, h_{3,0}, h_{3,1}, \widetilde{h}_{4,1}).$$

The parabola conjecture predicts the structure of $\pi_* \widehat{Z}$, and in particular predicts that the map (8.0.1) is neither injective nor surjective in homotopy.

8.1. **The localized Adams spectral sequence for Z.** Consider the localized Adams spectral sequence

$$v_2^{-1}\, {}^{ass}E_2^{*,*}(Z) \Rightarrow \pi_* \widehat{Z}. \tag{8.1.1}$$

The E_2-term of this spectral sequence was computed in Proposition 6.2.1:

$$v_2^{-1}\, {}^{ass}E_2^{*,*}(Z) \cong \mathbb{F}_2[v_2^{\pm}, \widetilde{h}_{2,1}, h_{3,0}, h_{3,1}, h_{4,0}, h_{4,1}, \ldots].$$

The analog of Mahowald-Ravenel-Shick's *differentials conjecture* [MRS01, Conj. 3.16] is the following.

Conjecture 8.1.2. (Differentials Conjecture) In the localized Adams spectral sequence (8.1.1) we have

$$\begin{aligned} d_2(h_{4,0}) &= v_2 \widetilde{h}_{2,1}^2, \\ d_2(h_{i,0}) &= v_2 h_{i-2,1}^2, \\ d_4(h_{i,1}) &= v_2 h_{i-1,0}^4. \end{aligned}$$

The idea is that the d_2 differentials in the above conjecture are lifted from the analogous differentials in the May-Ravenel spectral sequence (Theorem 5.2.2), and that the d_4 differentials arise from these through an extended power argument [Rav92b].

Assuming these are the only d_r differentials for $r \leq 4$, and that they satisfy the Leibniz rule,[10] we would have

$$v_2^{-1}\, {}^{ass}E_5^{*,*}(Z) \cong \mathbb{F}_2[v_2^{\pm}] \otimes E(\widetilde{h}_{2,1}, h_{3,0}, h_{3,1}, x_3, x_4, x_5, \cdots)$$

where[11]

$$x_i := h_{i,0}^2.$$

In the discussion after Conjecture 5.12 of [MRS01] (see also [Rav92b]), Mahowald-Ravenel-Shick predict the collapse of the localized ASS for $y(n)$ at a finite stage. The analog of their conjecture in our context is the following.

Conjecture 8.1.3 (Parabola Conjecture)**.** The localized ASS for Z collapses at E_5, and therefore

$$\pi_* \widehat{Z} \cong \mathbb{F}_2[v_2^{\pm}] \otimes E(\widetilde{h}_{2,1}, h_{3,0}, h_{3,1}, x_3, x_4, x_5, \cdots).$$

Moreover, the telescope conjecture is false, and the kernel of (8.0.1) is the ideal

$$(x_3, x_4, \cdots) \subset \pi_* \widehat{Z}$$

and the ideal

$$(\widetilde{h}_{4,1}) \subset \pi_* Z_{K(2)}$$

[10] Note that Z is *not* a ring spectrum, as we have already seen in the topological AKSS, where $\widetilde{h}_{2,1}$ is a permanent cycle but $\widetilde{h}_{2,1}^2$ supports a non-trivial differential.

[11] In particular, we have $h_{3,0}^2 = x_3$ rather than $h_{3,0}^2 = 0$, but this is somewhat irrelevant given that Z is not a ring spectrum. Our choice to present $v_2^{-1} E_5$ in this manner leads to a more uniform discussion.

maps isomorphically onto the cokernel of (8.0.1).

Remark 8.1.4. Note that the element $v_2^{-10}\widetilde{h}_{4,1}$ is the image of the element $\zeta_2 \in \pi_{-1}S_{K(2)}$ (see the proof of Theorem 7.5.1), so the second part of the parabola conjecture predicts that ζ_2 is not in the image of the telescopic homotopy. However, the collapsing of the localized Adams spectral sequence does *not* directly imply that ζ_2 is not in the image of the telescopic homotopy: ζ_2 could be detected by a v_2-family of lower Adams filtration than the v_2-family corresponding to $h_{4,1}$ in the localized ASS.

We will now explain why we call Conjecture 8.1.3 the "parabola conjecture."

8.2. **Unbounded v_2-torsion in the tmf-ASS for Z.** The key to Mahowald's proof of the telescope conjecture at chromatic level 1 was his *bounded torsion theorem* [Mah81], which states that the E_2-page of the bo-ASS for the sphere decomposes into a direct sum of v_1-periodic classes, and v_1^2-torsion classes. We will explain how the analogous phenomenon likely fails in the context of the tmf-ASS for Z.

We have already seen (Theorem 5.4.3) that the May-Ravenel E_1-page has unbounded v_2-torsion. But we must run some more differentials in the tmf-ASS to relate this unbounded v_2-torsion to the kernel of the map (8.0.1).

We will assume the following optimistic conjecture in order to simplify our discussion.

Conjecture 8.2.1 (Bounded Torsion Conjecture)**.** The May-Ravenel spectral sequence collapses at E_1 with no hidden v_2-extensions.

Then $H^{*,*}(\mathcal{C})$ has basis:

$$(\mathrm{I}') \quad v_2^m h_{3,0}^{\bar{\epsilon}_3}\widetilde{h}_{2,1}^{\epsilon_2}h_{3,1}^{\epsilon_3}\widetilde{h}_{4,1}^{\epsilon_4}, \qquad m \geq 0;\ \epsilon_j, \bar{\epsilon}_j \in \{0,1\},$$

$$(\mathrm{I}'') \quad v_2^{<2^{i+1}} h_{3,0}^{\bar{\epsilon}_3} x_i^{k_i+1} x_{i+1}^{k_{i+1}} x_{i+2}^{k_{i+2}} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3}\widetilde{h}_{4,1}^{\epsilon_4} h_{i+3,1}^{\epsilon_{i+3}} \cdots, \qquad i \geq 3;\ k_j \geq 0;\ \epsilon_j, \bar{\epsilon}_j \in \{0,1\},$$

$$(\mathrm{II}) \quad h_{3,0}^{\bar{\epsilon}_3} h_{i+3,0}^{\bar{\epsilon}_{i+3}} h_{i+4,0}^{\bar{\epsilon}_{i+4}} \cdots x_3^{k_3} x_4^{k_4} \cdots \widetilde{h}_{2,1}^{\epsilon_2} \cdots h_{i-1,1}^{\epsilon_{i-1}} h_{i,1}^{l_i+2} h_{i+1,1}^{l_{i+1}} \cdots, \qquad i \geq 2;\ k_j, l_j \geq 0;\ \epsilon_j, \bar{\epsilon}_j \in \{0,1\}.$$

The long exact sequence (3.3.2) implies that the unbounded v_2-torsion in ${}^{\mathrm{tmf}}E_2^{*,*}(Z)$ arises from the terms (I$'$) and (I$''$) above. Since the terms (II) above, as well as $H^{*,*}(V)$ are v_2^1-torsion, the elements of ${}^{\mathrm{tmf}}E_2^{*,*}(Z)$ not mapping to terms of the form (I$'$) or (I$''$) are at most v_2^2-torsion.

The d_4-differentials of the Differentials Conjecture 8.1.2 suggest the following analogous conjecture for the tmf-ASS.

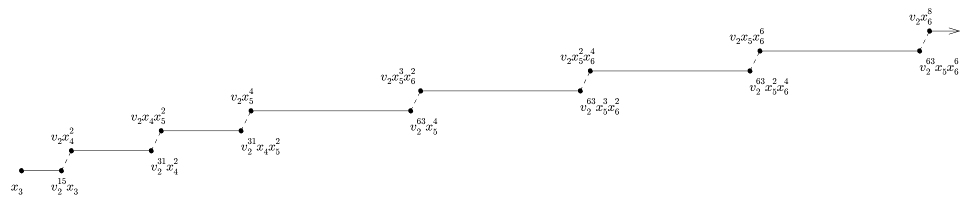

FIGURE 8.1. The v_2-periodic family supported by x_3

Conjecture 8.2.2 (Differentials Conjecture, v2, part 1)**.** In the tmf-ASS for Z there are differentials

$$\begin{aligned} &d_3(v_2^m h_{3,0}^{\bar{\epsilon}_3} x_i^{k_i+1} x_{i+1}^{k_{i+1}} x_{i+2}^{\bar{\epsilon}_{i+2}} x_{i+3}^{\bar{\epsilon}_{i+3}} \cdots x_{l-1}^{k_{l-1}} x_l^{k_l} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} h_{l,1} h_{l+1,1}^{\epsilon_{l+1}} h_{l+2,1}^{\epsilon_{l+2}} \cdots) \\ &\quad = v_2^{m+1} h_{3,0}^{\bar{\epsilon}_3} x_i^{k_i+1} x_{i+1}^{k_{i+1}} x_{i+2}^{\bar{\epsilon}_{i+2}} x_{i+3}^{\bar{\epsilon}_{i+3}} \cdots x_{l-1}^{k_{l-1}+2} x_l^{k_l} \cdots \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} h_{l+1,1}^{\epsilon_{l+1}} h_{l+2,1}^{\epsilon_{l+2}} \cdots \\ &\qquad + \cdots \end{aligned}$$

for $i \geq 3$, $l \geq i+3$, $m < 2^{i+1} - 1$, $k_j \geq 0$, and $\epsilon_j, \bar{\epsilon}_j \in \{0, 1\}$.

After running these d_3-differentials the only remaining classes in the tmf-ASS for Z are either v_2^2-torsion, or of the form

$$(\mathrm{I}') \quad v_2^m h_{3,0}^{\bar{\epsilon}_3} \widetilde{h}_{2,1}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4}, \qquad m \geq 0;\ \epsilon_j, \bar{\epsilon}_j \in \{0,1\},$$

$$(\mathrm{I}''') \quad v_2^{<2^{i+1}} \widetilde{h}_{2,1}^{\epsilon_1} h_{3,0}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} x_i^{k_i+1} x_{i+1}^{k_{i+1}} x_{i+2}^{\bar{\epsilon}_{i+2}} x_{i+3}^{\bar{\epsilon}_{i+3}} \cdots, \qquad i \geq 3;\ k_j \geq 0;\ \epsilon_j, \bar{\epsilon}_j \in \{0,1\}.$$

8.3. **Parabolas.** In the tmf-ASS for Z, we have differentials (Theorem 5.2.2)

$$d_1(h_{i+2,1}) = v_2^{2^{i+1}} x_i$$

whereas in the Adams spectral sequence there are conjecturally differentials (Conjecture 8.1.2)

$$d_4(h_{i+2,1}) = v_2 x_{i+1}^2.$$

This suggests the following.

Conjecture 8.3.1 (Extension Conjecture)**.** In the tmf-ASS there are hidden extensions

$$v_2^{2^{i+1}} x_i = v_2 x_{i+1}^2.$$

This conjecture predicts that the v_2-torsion families of type (I$'''$) of Conjecture 8.2.2 in the tmf-ASS for Z actually assemble via an infinite sequence of hidden extensions to form v_2-periodic families in $\pi_*(Z)$. Figure 8.1 shows one such v_2-periodic family.

If we assign a mass to x_i via

$$M(x_i) := \frac{1}{2^{i-2}}$$

and, more generally for monomials

$$M(v_2^m x_3^{k_3} x_4^{k_4} \cdots) := k_3 M(x_3) + k_4 M(x_4) + \cdots$$

then one finds that all of the terms of the form

$$v_2^m x_i^{k_i+1} x_{i+1}^{k_{i+1}} x_{i+2}^{\bar{\epsilon}_{i+2}} x_{i+3}^{\bar{\epsilon}_{i+3}} \cdots$$

(for $i \geq 3$, $0 < m < 2^{i+1}$, $k_j \geq 0$, and $\bar{\epsilon}_j \in \{0,1\}$) lie in the same v_2-periodic family if and only if they have the same mass.

Each of these v_2-periodic families begins with a term of the form

$$x_3^{k_3} x_4^{\bar{\epsilon}_4} x_5^{\bar{\epsilon}_5} \cdots$$

(with $k_3 \geq 0$ and $\bar{\epsilon}_j \in \{0,1\}$) with corresponding mass

$$M = \frac{k_3}{2} + \frac{\bar{\epsilon}_4}{4} + \frac{\bar{\epsilon}_5}{8} + \cdots .$$

Thus for each monomial

$$\widetilde{h}_{2,1}^{\epsilon_1} h_{3,0}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} \in E(\widetilde{h}_{2,1}, h_{3,0}, h_{3,1}, \widetilde{h}_{4,1})$$

and each mass $M \in \mathbb{Z}[1/2]^{>0}$ there is a corresponding non-trivial monomial

$$x_3^{k_3} x_4^{\bar{\epsilon}_4} x_5^{\bar{\epsilon}_5} \cdots \in \mathbb{F}_2[x_3] \otimes E(x_4, x_5, x_6, \cdots)$$

such that

$$\widetilde{h}_{2,1}^{\epsilon_1} h_{3,0}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} x_3^{k_3} x_4^{\bar{\epsilon}_4} x_5^{\bar{\epsilon}_5} \cdots$$

supports a v_2-family with mass M. For each of these v_2-families, the elements

$$\widetilde{h}_{2,1}^{\epsilon_1} h_{3,0}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}^{\epsilon_4} v_2 x_i^{2^{i-2}M}$$

represent a cofinal collection of elements which lie in the family. The elements $v_2 x_i^{2^{i-2}M}$ lie on the (sideways) parabola

$$t - n = \frac{4}{M} n^2 - 3n + 6 \tag{8.3.2}$$

in the $(t-n, n)$-plane. As such, we will refer to these v_2-families as v_2-*parabolas*.

8.4. **The vanishing line.** Theorem 5.4.3 and Proposition 6.2.4 imply the following.

Theorem 8.4.1. *In the* tmf*-ASS for Z, we have ${}^{\mathrm{tmf}}E_2^{n,t}(Z) = 0$ for*

$$n > \frac{t-n+12}{11}.$$

Unfortunately, Conjecture 8.2.1 only predicts the bounded v_2-torsion in this E_2-term is v_2^2-torsion. This means that the v_2^2-torsion could in principle assemble (via infinite sequences of hidden extensions) to detect non-trivial v_2-periodic families in $\pi_* Z$ which lie along curves with derivatives $\geq 1/12$ in the $(t-n, n)$-plane. Thus Theorem 8.4.1 is not strong enough to preclude the bounded v_2^2-torsion contributing to the homotopy of $\widehat{Z}$.

This $1/11$ vanishing line essentially arises from the element $\widetilde{h}_{2,1} \in H^{*,*}(\mathcal{C})$.[12] However, the results of [AD73] imply that ${}^{ass}E_2^{*,*}(A_2)$ has a vanishing line of slope $1/13$. Moreover, the element $h_{2,2}^4$ in the May spectral sequence (corresponding to $\widetilde{h}_{2,1}^4 \in H^{*,*}(\mathcal{C}(Z))$) detects the element $g_2 \in {}^{ass}E_2(S)$. The element g_2 is not nilpotent [Isa14], but it detects the element $\bar{\kappa}_2 \in \pi_{44}(S)$ which necessarily is nilpotent by the Nishida nilpotence theorem. It seems likely this can be used to prove the following, which would imply that the bounded v_2^2-torsion cannot contribute to the homotopy of $\widehat{Z}$.

Conjecture 8.4.2 (Vanishing Line Conjecture)**.** There is an r so that ${}^{\mathrm{tmf}}E_r^{n,t}(Z)$ has a $1/13$ vanishing line.

8.5. **The parabola conjecture.** Assuming all of the conjectures so far are true, the homotopy of $\widehat{Z}$ can only be detected by the v_2-periodic elements or the v_2-parabolas in ${}^{\mathrm{tmf}}E_4(Z)$. We therefore are left to consider the possibility of differentials between these families. The only possibilities are:

(1) differentials between v_2-periodic elements,
(2) differentials from v_2-periodic elements to v_2-parabolas,
(3) differentials from a v_2-parabola of mass M to a v_2-parabola of mass M' with $M' > M$.

Differentials of type (1) are ruled out by Theorem 7.5.1. Proposition 7.4.4 establishes that $\widetilde{h}_{2,1}$, $h_{3,0}$, and $v_2^2 h_{3,1}$ are permanent cycles in the tmf-ASS. While Z is not a ring spectrum, one might nevertheless suspect that the v_2-families

$$v_2^m \widetilde{h}_{2,1}^{\epsilon_1} h_{3,0}^{\epsilon_2} h_{3,1}^{\epsilon_3}$$

cannot support differentials of type (2), and presumably this could be easily established be extending our low dimensional calculations a little further.

We therefore turn to considering differentials of type (2) involving the element $\widetilde{h}_{4,1}$. Note that since $v_2^{-10}\widetilde{h}_{4,1}$ detects $\zeta_2 \in \pi_{-1} Z_{K(2)}$, this is equivalent to the question of whether the element $\zeta_2 \in \pi_{-1} Z_{K(2)}$ lifts to $\pi_{-1}\widehat{Z}$ (compare with Remark 8.1.4).

We first note that Conjecture 8.2.2 does not include the differential

$$d_4(h_{4,1}) = v_2 x_3^2$$

of Conjecture 8.1.2. We therefore offer this second installment to Conjecture 8.1.2 which does include this differential, and its consequences.

Conjecture 8.5.1 (Differentials conjecture, v2, part 2)**.** In the tmf-ASS, for $m \gg 0$, the v_2-families

$$v_2^m \widetilde{h}_{2,1}^{\epsilon_1} h_{3,0}^{\epsilon_2} h_{3,1}^{\epsilon_3} \widetilde{h}_{4,1}$$

support differentials which hit the v_2-parabolas supported by

$$\widetilde{h}_{2,1}^{\epsilon_1} h_{3,0}^{\epsilon_2} h_{3,1}^{\epsilon_3} x_3^2$$

[12] If one replaces Z with the Thom spectrum $y(2)$ of [MRS01], a similar analysis to Theorem 8.4.1 easily yields a vanishing line of slope $1/13$.

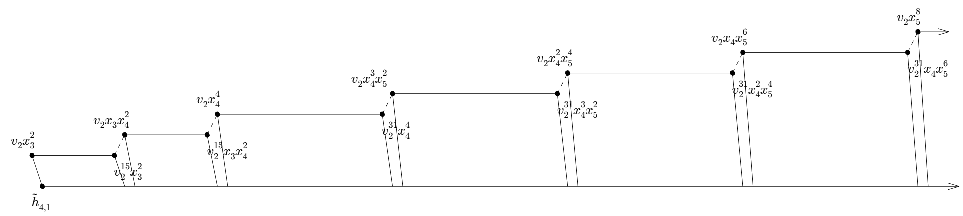

FIGURE 8.2. The conjectural differentials on $v_2^m\widetilde{h}_{4,1}$.

and the v_2-parabolas supported by

$$\widetilde{h}_{2,1}^{\epsilon_1}h_{3,0}^{\epsilon_2}h_{3,1}^{\epsilon_3}\widetilde{h}_{4,1}x_3^{k_3}x_4^{\bar{\epsilon}_4}x_5^{\bar{\epsilon}_5}\cdots$$

support differentials which hit the v_2-parabolas

$$\widetilde{h}_{2,1}^{\epsilon_1}h_{3,0}^{\epsilon_2}h_{3,1}^{\epsilon_3}x_3^{k_3+2}x_4^{\bar{\epsilon}_4}x_5^{\bar{\epsilon}_5}\cdots.$$

Figure 8.2 shows an example of such a family of differentials. Note that the lengths of each of the families of differentials predicted by Conjecture 8.5.1 are unbounded. However, it could be that far enough out in the family, the differentials are all zero. This could occur, for instance, if another parabola supporting shorter differentials kills the v_2-family which is the putative target. Such a phenomenon would be a means for ζ_2 to exist in $\pi_*\widehat{Z}$ without violating Conjecture 8.5.1.

The following version of the parabola conjecture offers a maximally anti-telescope point of view, and is consistent with Conjecture 8.1.3.

Conjecture 8.5.2 (Parabola Conjecture, v2)**.** The differentials of Conjecture 8.5.1 are non-trivial, and all of the remaining v_2-parabolas have elements which are permanent cycles. Thus the v_2-periodic homotopy of Z is generated by the v_2-families

$$v_2^m h_{3,0}^{\bar{\epsilon}_3}\widetilde{h}_{2,1}^{\epsilon_2}h_{3,1}^{\epsilon_3}, \quad m \geq 0,\ \epsilon_j \in \{0,1\},$$

and the v_2-parabolas supported by

$$h_{3,0}^{\bar{\epsilon}_3}\widetilde{h}_{2,1}^{\epsilon_2}h_{3,1}^{\epsilon_3}x_3^{\bar{\epsilon}_3}x_4^{\bar{\epsilon}_4}\cdots, \quad \epsilon_j, \bar{\epsilon}_j \in \{0,1\}.$$

Remark 8.5.3. Some recent work of Carmeli-Schlank-Yanovski seems to actually imply that $\zeta_2 \in \pi_*Z_{K(2)}$ lifts to an element of $\pi_*\widehat{Z}$. If this turns out to be true, than it flies in the face of the conventional wisdom on the subject, but it does not seem to necessarily force the telescope conjecture to be true. Rather, it is totally possible that a weak form of the parabola conjecture is true, where the map

$$\pi_*\widehat{Z} \to \pi_*Z_{K(2)}$$

is surjective with non-trivial kernel generated by a portion of the v_2-parabolas.

Appendix A. $A(2)$ as a module over the Steenrod algebra

Here, we describe the A-module structure on $A(2)$ resulting from [Rot77, p. 30, Chapter III] and present it as a definition file for Bruner's program [Bru93]. The definition file is a text file, where the first line is an integer n which records the dimension of the A-module as an $\mathbb{F}_2$-vector space. We should then interpret that we are given an ordered basis $g_0, \ldots, g_{n-1}$. The second line of the text file is an ordered list of integers $d_0, \ldots, d_{n-1}$, where d_i is the internal degree of g_i. For $A(2)$, the first two lines of Bruner's definition file reads as:

```
64

0 1 2 3 3 4 4 5 5 6 6 6 7 7 7 7 8 8 8 9 9 9 9 10 10 10 10 10 11 11 11
11 12 12 12 12 13 13 13 13 13 14 14 14 14 15 15 15 16 16 16 16 17 17
17 18 18 19 19 20 20 21 22 23
```

Every subsequent line in the text file describes a nontrivial action of some Sq^k on some generator g_i. For example, if

$$Sq^k(g_i) = g_{j_1} + \ldots + g_{j_l}$$

we would encode this fact by writing the line

$$i\ k\ l\ j_1\ \ldots\ j_l$$

followed by a line break. Actions which are not indicated by such data are assumed to be trivial.

0 1 2 3
3 4 4 5
5 6 6 6
7 7 7 7
8 8 8 9
9 9 9 10
10 10 10 10
11 11 11 11
12 12 12 12
13 13 13 13
13 14 14 14
14 15 15 15
16 16 16 16
17 17 17 18
18 19 19 20
20 21 22 23

0 1 1 1
0 2 1 2
0 3 1 3
0 4 1 5
0 5 1 7
0 6 1 9
0 7 1 12
0 10 1 23
0 12 1 32
0 13 1 36
0 14 1 41
0 20 2 59 60
0 21 1 61

1 2 2 3 4
1 3 1 6
1 4 2 7 8
1 5 1 10
1 6 2 12 13
1 7 1 16
1 8 1 19
1 9 1 23
1 12 1 36
1 14 1 45
1 15 1 48
1 20 1 61
1 22 1 63

2 1 1 3
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2 5 2 12 14
2 6 2 16 17
2 7 1 20
2 8 1 24
2 9 1 28
2 10 1 32
2 11 1 36
2 12 2 41 42
2 14 1 49
2 15 1 52
2 16 1 55
2 18 2 59 60
2 19 1 61

3 2 1 8
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3 4 2 12 14
3 6 3 19 20 21
3 7 2 23 25
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4 4 2 13 15
4 5 2 16 18
4 6 2 19 22
4 7 2 23 26
4 8 1 30
4 9 1 34
4 10 1 38
4 11 1 42
4 12 1 45
4 13 1 48
4 14 1 52
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5 8 2 34 35
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5 10 2 42 43
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5 13 1 52
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5 18 1 62
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6 8 2 34 35
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40 4 1 53
40 5 1 55
40 6 1 57
40 7 1 59
40 10 1 63

41 2 1 48
41 4 1 55
41 6 2 59 60
41 7 1 61
41 8 1 62
41 9 1 63

42 2 1 49
42 3 1 52
42 4 1 55
42 6 2 59 60
42 7 1 61
42 8 1 62
42 9 1 63

43 1 1 46
43 2 1 49
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43 3 1 52
43 4 1 56
43 5 1 58

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44 3 1 54
44 4 2 55 56
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62 1 1 63
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University of Colorado at Boulder

E-mail address: agnes.beaudry@colorado.edu

University of Notre Dame

E-mail address: mbehren1@nd.edu

University of Virginia

E-mail address: pb9wh@virginia.edu

University of Illinois at Urbana-Champaign

E-mail address: dculver@illinois.edu

Massachusetts Institute of Technology

E-mail address: xuzhouli@mit.edu