MATH 402 Worksheet 1

Friday 1/26/18

Exercise 1. We are studying the system of a small experimental school, which would like to implement the following rules in the new semester:

(A) There will be 5 offered courses: art, biology, history, math, sociology.
(B) To maximize focus, every student will take exactly two courses.
(C) To maximize interaction among students, every pair of courses will have exactly one common student.

Call this "axiomatic system" S_5. Find a way to geometrically model this system, and then answer the following questions (and prove your answers using only the given axioms (A)-(C)).

(1) For the new scheme to work, how many students ought to be enrolled in the school?
(2) Show that any two students can have at most one course in common.
(3) How many students are in each class?
(4) How would the answers to the previous questions change if there are n offered courses, where n is an arbitrary natural number? Call such a system S_n.
(5) Make a sketch for S_3, S_4, S_5.

Exercise 2. Consider the following set of axioms:

(1) Every student is enrolled in at least two classes,
(2) Every class has at least two students enrolled in it,
(3) There is at least one student.

Answer the following questions.

(a) What are the undefined terms?
(b) Prove the following theorem: there is at least one class (this is not stated in the axioms, and so must be proved.)
(c) What is the minimum number of classes? To answer this, you must need to find a model with a certain number of classes, and must argue that any other model can't have any fewer.
(d) Find two non-isomorphic models.
(e) Define each of the terms and discuss whether or not they apply to this axiomatic system:
    - Consistent,
    - Categorical,
    - Complete
(f) What does it mean for an axiom to be *independent*? Show that the second axiom is independent of the other two.

Exercise 3. Consider the axiomatic system where the undefined terms consist of elements of a set S, and a set P consisting of certain pairs of elements of S. (So an element of P looks like (a, b), where a, b are in S, but not all pairs (a, b) are elements of P.) The axioms are as follows:

(1) If (a, b) is in P, then (b, a), is not in P.
(2) If (a, b) and (b, c) are in P, then (a, c) is in P.

a Let $S_1 = \{1, 2, 3, 4\}$, and let $P_1 = \{(1, 2), (2, 3), (1, 3)\}$. Is this a model for the system? (Justify your answer, always justify your answers, unless specifically told otherwise.)

b Let $S_2 = \mathbb{R}$,the set of all real numbers. Let $P_2 = \{(x, y)|x < y\}$. Is this a model for the system?

c Use this to argue that the axiomatic system is not complete. In particular, can you add an axiom such that (S_1, P_1) is a model for the new system, but (S_2, P_2) is not?

You do not need to turn in this worksheet. However, you should make sure to work out the problems. The problems that appear on worksheets may appear on the examines, so be sure to work out solutions even if you don't finish in class. Come to office hours if you have questions.