MATH 402 Worksheet 5

Friday 3/2/18

Exercise 1.

(1) Recall the definition of a group.
(2) Let G denote the set of isometries of the plane. Show that G is a group. Is it abelian?

Exercise 2.

Definition 0.1. Let f be an isometry. A point P is a *fixed point* of f if $f(P) = P$.

(1) Show that if A and B are fixed points of an isometry f, then f fixes every point on the line formed by A and B.
(2) Let A, B, C be three non-collinear points which are fixed by f. Argue that f is the identity map.
(3) As a corollary, argue that if f and g are two isometries which agree on three points, then $f = g$.

Exercise 3.

(1) Let ℓ be a line. Give a geometric description of the reflection about the line ℓ. Is it an isometry? What are its fixed points?
(2) Conversely, suppose that f is an isometry which fixes the points on a line ℓ and no others. Argue that f is a reflection.

Exercise 4.

(1) Make the identification of the Euclidean plane with the Cartesian plane $\mathbb{R}^2$. Recall that $\mathbb{R}^2$ is a two dimensional real vector space. Suppose that f is an isometry that fixes the origin. Show that f is a linear map $\mathbb{R}^2 \to \mathbb{R}^2$.
(2) Show that the set of isometries fixing the origin form a group.
(3) Show that a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry if and only if f preserves the inner product

$$(f(u)) \cdot (f(v)) = u \cdot v.$$

(4) This group is also known as the *two dimensional orthogonal group*, denoted $O(2)$, $O_2(\mathbb{R})$, or $O(2, \mathbb{R})$. It can be identified with the set of 2-by-2 matrices M with $M \cdot M^t = 1$.