

MATH 402 Worksheet 7

Friday 4/6/18

Exercise 1. Recall that we defined in class what it meant for a ray Aa to be a *limiting parallel* of a ray Bb , which we will denote by $Aa \parallel Bb$. In this exercise, you will sketch out the proof that this defines an equivalence relation.

- (1) Show that the relation is symmetric, i.e. if $Aa \parallel Bb$ then $Bb \parallel Aa$. You will need to rotate and use Pasch's axiom to make this argument. (Drawing a picture may be helpful)
- (2) Show that this relation is transitive, i.e. if $Aa \parallel Bb$ and $Bb \parallel Cc$ then $Aa \parallel Cc$. To prove this, first show that if a, b, c are distinct lines, then we can make A, B and C to be collinear.

Since the relation of being a limiting parallel defines an equivalence relation, we can make the following definition

Definition 1. An *omega point* is an equivalence class of rays under the relation of being limiting parallels. A *generalized point* will refer to either a point in the usual sense or an omega point.

Thus, given a line ℓ , and points P, Q on ℓ , there is an omega point Ω determined by the ray \overrightarrow{PQ} .

Exercise 2.

- (1) Show that any line ℓ determines two distinct omega points. That there are two arises from the fact that, once a point P is fixed on ℓ , there are two sides of ℓ (this is what Hvidsten confusingly refers to as left and right).
- (2) We will say that two lines ℓ and m intersect at an omega point if they share a common omega point. Give a more detailed mathematical definition of this concept.
- (3) Show that lines ℓ and m intersect at an omega point if and only if ℓ is a limiting parallel of m .
- (4) Show that given any point P and any omega point ω , there is a unique line ℓ passing through P and ω .
- (5) Show that if ω and ω' are distinct omega points, then there is a unique line connecting them.
- (6) Given concrete descriptions of omega points for the Klein and Poincaré disc models.
- (7) Draw pictures describing triangles where
 - (a) one of the vertices is an omega point
 - (b) two of the vertices are omega points
 - (c) all of the vertices are omega points

Definition 2. The *extended hyperbolic plane* will consist of the points of the hyperbolic plane and the omega points of the plane.

Exercise 3. Let f be an isometry of the hyperbolic plane. Show that f naturally extends to the extended hyperbolic plane.