CHROMATIC HOMOTOPY THEORY

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CONTENTS

1. The Classical Adams spectral sequence

In this section of the notes we will give an overview of the classical Adams spectral sequence. Namely, we will discuss what it is as well as what it calculates. We will briefly discuss issues of convergence and describe its E_2-term.

1.1. The Construction.

Throughout this section, we will let H denoted $H\mathbb{F}_p$ for some prime p.

Question 1.1. Given a spectrum X, how can we compute its homotopy groups $\pi_* X$? Can we do this starting with knowledge about $H^* X$?

Recall that if X is a spectrum, then H^*X is a module over the Steenrod algebra A. Indeed, suppose $\vartheta \in A_*$ is a stable cohomology operation, and $x \in H^*X$ is a cohomology class, then x is represented by a morphism $X \to \Sigma^n H$ and ϑ by a morphism $\vartheta : H \to \Sigma^k H$. Thus, we obtain a new map $\vartheta \cdot x$ by

$$X \xrightarrow{x} \Sigma^n H \xrightarrow{\vartheta} \Sigma^{n+k} H.$$

Since the action by A is defined by post-composition, it gives a left action of A on H^*X. Thus, we have

Proposition 1.2. *The functor H^* takes values in graded left A-modules.*

Here is the idea of the Adams spectral sequence. Suppose we have a spectrum X and we have a homotopy class $\alpha : S^n \to X$. How could we tell if α is non-trivial? Well, if the composite

$$S^n \xrightarrow{\alpha} X \simeq S^0 \wedge X \xrightarrow{\eta \wedge X} H \wedge X$$

is non-trivial then α is nontrivial. Note that the second morphism is the Hurewicz map

$$h : \pi_* X \to H_* X.$$

However, the Hurewicz map will fail to see most elements in $\pi_* X$.

Example 1.3. If $\alpha : S^0 \to S^0$ is the degree n map, then $h(f) \neq 0$ if and only if $(n, p) = 1$. However, we know that $\pi_* S^0$ is non-zero in infinitely many degrees.

On the other hand, if $h(\alpha) = 0$, then α lifts to a map $\tilde{\alpha} : S^n \to \Sigma^{-1}\overline{H} \wedge X$. Here, $\overline{H}$ is the cofibre of the unit map $\eta : S \to H$. Thus, we have built a diagram

$$\begin{array}{ccc} S^n & & \\ & \searrow^{\alpha} \quad \searrow^{\tilde{\alpha}} & \\ & X & \longleftarrow \Sigma^{-1}\overline{H} \wedge X \\ & \downarrow & \downarrow \\ & H \wedge X & H \wedge \overline{H} \wedge X. \end{array}$$

The morphism α is nonzero provided $\tilde{\alpha}$ is non-zero To check that $\tilde{\alpha}$ is non-trivial, we consider the composite

the horizonal map could be zero... say better.

$$S^n \xrightarrow{\tilde{\alpha}} \Sigma^{-1}\overline{H} \wedge XH \longrightarrow \Sigma^{-1}H \wedge \overline{H} \wedge X.$$

IF this is nonzero, then $\tilde{\alpha}$ is non-zero, and so α is nonzero. If, on the other hand, it is zero, then we repeat the process. This suggests the following diagram,

$$\begin{array}{ccccccc} X & \longleftarrow & \Sigma^{-1}\overline{H} \wedge X & \longleftarrow & \Sigma^{-2}\overline{H}^{\wedge 2} \wedge X & \longleftarrow & \cdots \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\ H \wedge X & & \Sigma^{-1}H \wedge \overline{H} \wedge X & & \Sigma^{-2}H \wedge \overline{H}^{\wedge 2} \wedge X & & \end{array}.$$

Applying the functor π_* yields an exact couple with

$$\begin{array}{ccccccc} \pi_* X & \longleftarrow & \pi_*\Sigma^{-1}\overline{H} \wedge X & \longleftarrow & \pi_*\Sigma^{-2}\overline{H}^{\wedge 2} \wedge X & \longleftarrow & \cdots \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\ H_* X & & \pi_*\Sigma^{-1}H \wedge \overline{H} \wedge X & & \pi_*\Sigma^{-2}H \wedge \overline{H}^{\wedge 2} \wedge X & & \end{array}.$$

More explicitly, we have an exact couple with $D_1^{s,t} := \pi_{t-s}\overline{H}^{\wedge s} \wedge X$ and $E_1^{s,t} := \pi_{t-s}H \wedge \overline{H}^{\wedge s} \wedge X$.

So we have produced a spectral sequence! But there are two obvious questions that immediately arise.

Question 1.4.

(1) Does this spectral sequence converge, and if so, to what?
(2) Can we give a nice description of the E_2-term?

Thus far, we have actually constructed something called the *canonical Adams resolution*. We would like to have a more general notion of an Adams resolution. First, let me state the theorem.

Theorem 1.5. *[Adams] Let X be a spectrum, then there is a spectral sequence $E_r^{s,t}(X)$ with differentials*

$$d_r : E_r^{s,t}(X) \to E_r^{s+r,t+r-1}(X)$$

such that

(1) $E_2^{s,t} \cong \mathrm{Ext}_A(H^*X, \mathbb{F}_p)$,
(2) *if X is a spectrum of finite type, then this spectral sequence converges strongly (in the sense of Boardmann) to $\pi_*(X) \otimes \mathbb{Z}_p$.*

We need the following standard facts.

Proposition 1.6. *The following are true.*

- $H_*X \cong \pi_*(H \wedge X)$,
- $H^*X \cong [X, H]_*$,
- $H^*H = A$,
- *If $K = \bigvee_\alpha \Sigma^{|\alpha|}H$ is a locally finite wedge of suspensions of Eilenberg-MacLane spectra, then*
$$\pi_*K \cong \bigoplus_\alpha \Sigma^{|\alpha|}\mathbb{F}_p \cong \mathrm{hom}_A(H^*K, \mathbb{F}_p).$$
- *If K is as above, then a map $f : X \to K$ determines a locally finite collection $\{f_\alpha : X \to \Sigma^{|\alpha|}H\}_\alpha$. Conversely, any locally finite collection in H^*X determines a map to such a generalized EM spectrum.*
- *If $\{x_\alpha\}_\alpha \subseteq H^*X$ is a locally finite collection which generates H^*X as an A-module, then the corresponding map $f : X \to K$ is a surjection in cohomology,*
- *$H \wedge X$ is a generalized EM spectrum, it has a suspension of H for each element of a basis for H^*X. Also,*
$$H^*(H \wedge X) \cong A \otimes H^*X,$$
and the map
$$X \simeq S^0 \wedge X \to H \wedge X$$
induces the multiplication map
$$A \otimes H^*X \to H^*X$$
in cohomology.

Remark 1.7. If V is a graded $\mathbb{F}_p$ vector space, then we usually let HV denote the generalized EM spectrum with $\pi_*HV = V$.

Definition 1.8. A *mod p Adams resolution* of a spectrum X is a collection of spectra X_s for each $s \in \mathbb{N}$ and maps $g_s : X_{s+1} \to X_s$ such that the following conditions hold:

- $X_0 = X$,
- If $K_s := \mathrm{cofib}(g_s)$, then K_s is a generalized EM-spectrum for some mod p vector space,
- The induced map $f_s : X_s \to K_s$ is a surjection in $H^*(-)$.

Remark 1.9. We usually express this data diagrammatically as follows.

$$\begin{array}{ccccccc} X = X_0 & \xleftarrow{g_0} & X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{g_2} & \cdots \\ \downarrow f_0 & \overset{\partial}{\nearrow} & \downarrow f_1 & \overset{\partial}{\nearrow} & \downarrow f_2 & \overset{\partial}{\nearrow} & \\ K_0 & & K_1 & & K_2 & & \end{array}$$

where the hooks are cofibre sequences, and the dashed arrows are the connecting maps. In particular, $\partial : K_s \to \Sigma X_{s+1}$.

Exercise 1. Show that canonical Adams resolution produced above is in fact an example of an Adams resolution in the above sense.

Observation 1. There are several things to observe at this point. Note first that since each K_s is a generalized EM spectrum on some mod p vector space, then H^*K_s is a free A-module. Moreover, since f_s induces a surjection in cohomology, the long exact sequence of the cofibre sequence

$$X_{s+1} \xrightarrow{g_s} X_s \xrightarrow{f_s} K_s$$

degenerates into the following short exact sequence

$$0 \longrightarrow H^*(\Sigma X_{s+1}) \xrightarrow{\delta} H^*K_s \xrightarrow{H^*f_s} H^*X_s \longrightarrow 0$$

We can splice together these short exact sequence together to form an exact sequence in the following manner.

(1.10)

$$\begin{array}{ccccccccc} & & & & H^*(\Sigma X_1) & & & & \\ & & & \swarrow & & \nwarrow & & & \\ 0 \longleftarrow & H^*(X) & \longleftarrow & H^*(K_0) & \longleftarrow & H^*(\Sigma K_1) & \longleftarrow & & H^*(\Sigma^2 \cdots \\ & & & & & \nwarrow & & \swarrow & \\ & & & & & & H^*(\Sigma^2 X_2) & & \end{array}$$

Since each of the $H^*(K_s)$ is a free A-module, this is a projective resolution of H^*X in A-modules.

Exercise 2. Show that any free resolution of H^*X in A-modules arises in this manner from an Adams resolution.

Remark 1.11. This is, in fact, the way Adams originally thought about Adams resolutions. If you look at the original paper, [1], Adams is really just trying to build spectrum level version of a free resolution of H^*X. However, the language in that paper is a little clunky from the modern perspective as it is before spectra became mainstream. Thus, Adams is trying to build the resolution in spaces, but he can only do so in a range.

Now from an Adams resolution, we obtain an exact couple by setting $D_1^{s,t} := \pi_{t-s}(X_s)$ and $E_1^{s,t} := \pi_{t-s}(K_s)$. This gives an exact couple

$$\begin{array}{ccc} D_1^{**} & \xrightarrow{\quad i_1 \quad} & D_1^{**} \\ & {}_{k_1}\nwarrow \quad \swarrow_{j_1} & \\ & E_1^{**} & \end{array}$$

where

$$i_1 := \pi_{t-s}(g_s) : D_1^{s+1,t+1} = \pi_{t-s}(X_s+1) \to \pi_{t-s}(X_s) = D_1^{s,t}$$
$$j_1 = \pi_{t-s}(f_s) : D_1^{s,t} = \pi_{t-s}(X_s) \to \pi_{t-s}(K_s) = E_1^{s,t}$$

and

$$k_1 = \pi_{t-s}\partial : E_1^{s,t} = \pi_{t-s}(K_s) \to \pi_{t-s-1}(X_{s+1}) = D_1^{s+1,t}$$

This of course produces a spectral sequence $\{E_r^{s,t}(X), d_r\}$ in the usual manner. Let's now figure out the direction of the differentials.

Proposition 1.12. *The differentials in the spectral sequence constructed above are of the form*

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

Remark 1.13. I will not give a completely rigorous argument, as that would require me to write down far more diagrams then I wish to write. The reader can find a more rigorous argument in [15].

Proof. We have the following diagram, where $u = t - s$.

$$\begin{array}{ccccccc} \pi_u(X) & \xleftarrow{\pi_u g_0} & \pi_u X_1 & \xleftarrow{\pi_u g_1} & \pi_u X_2 & \xleftarrow{\pi_u g_2} & \cdots \\ \downarrow{\scriptstyle \pi_u f_0} & \overset{\partial}{\nearrow} & \downarrow{\scriptstyle \pi_u f_1} & \overset{\partial}{\nearrow} & \downarrow{\scriptstyle \pi_u f_2} & \overset{\partial}{\nearrow} & \\ \pi_u K_0 & & \pi_u K_1 & & \pi_u K_2 & & \end{array}.$$

Suppose that $x \in \pi_u(K_s)$ is an element which represents a class in $E_r^{s,t}$ of our spectral sequence. Thus, in particular, we have that $d_\ell(x) = 0$ for all $\ell < r$. The way an exact couple works, is that if this holds, then the element $\partial x \in \pi_{u-1}(X_{s+1})$ lifts to $\pi_{u-1}(X_{s+r})$. Let $\widetilde{\partial x}$ be such a lift. Then $d_r[x]$ is defined to be the projection of $\pi_{u-1}(f_{s+r})(\widetilde{\partial x})$ down to

E_r. Now note that $\pi_{u-1}(f_{s+r})(\widetilde{\partial x}) \in \pi_{u-1}(K_{s+r})$, and so contributes to $E_r^{s+r,t'}$, where $t' - (s+r) = u - 1 = t - s - 1$. Thus $t' = t + r - 1$. So $d_r[x] \in E_r^{s+r,t+r-1}$, as desired. □

Remark 1.14. Here is a diagrammatic way of writing this.

$$
\begin{array}{ccc}
\pi_{t-s-1}(X) & \longleftarrow & \pi_{t-s-1}(X_{s+r}) \\
 & & \downarrow \\
 & & \pi_{t-s-1}(K_{s+r}) \\
 & & \subseteq\uparrow \\
 & & Z_r \\
 & & \downarrow \\
 & & E_r^{s+r,t+r-1}
\end{array}
$$

put double head in last arrow. and finish diagram.

Remark 1.15. In this context we usually draw the spectral sequences using the *Adams indexing convention*. That means that instead of drawing spectral sequences with (t,s) indexing, we instead draw them with the indexing $(t-s,s)$. Then differentials d_r always go to the left 1 and go up r. So there is a direct relationship between the length of the differential and what page it is on.

draw picture...

Observation 2. Note that, by contruction, that $E_r^{s,t} = 0$ if $s < 0$. Thus, any $E_1^{s,t}$ can receive only a finite number of differentials. This allows us to make the following identification,

$$E_\infty^{s,t} = \bigcap_{r>s} E_r^{s,t}.$$

1.2. **The E_2-term.** Let's figure out what the E_2-term is. Let X be a spectrum and let (X_s, g_s) be an Adams resolution of X. Recall from the previous subsection, (1.10), that the cofibres K_s give us a resolution of H^*X:

$$0 \leftarrow H^*X \leftarrow H^*K_0 \leftarrow H^*(\Sigma K_1) \leftarrow H^*(\Sigma^2 K_s) \leftarrow \cdots .$$

Since the K_s are generalized Eilenberg-MacLane spectrum on some $\mathbb{F}_p$-vector space, it follows from Proposition 1.6 that H^*K_s are free modules over the Steenrod algebra. Since the above is an exact sequence, it follows that this is a free resolution of H^*X. We need to relate this to the E_1-term somehow.

The first step in relating this to the E_1-term is by using Proposition 1.6 to observe that

$$\pi_* K_s \cong \hom_A(H^*K_s, \mathbb{F}_p).$$

Note that the d_1-differential is given by the composite

$$d_1 = \pi_*(f_s \circ \partial) : \pi_* K_s \to \pi_* \Sigma K_{s+1}.$$

So we must relate these maps to the maps in the resolution (1.10). It is clear from the construction of the resolution that d_1-differential is the dual of maps in the resolution. Thus, the E_1-term can be equivalently expressed as applying the functor $\hom_A(-, \mathbb{F}_p)$ to the free resolution:

$$\hom_A(H^*K_0, \mathbb{F}_p) \to \hom_A(H^*\Sigma K_1, \mathbb{F}_p) \to \hom_A(H^*\Sigma^2 K_2, \mathbb{F}_p) \to \hom_A(H^*\Sigma^3 K_3, \mathbb{F}_p) \to \cdots.$$

Thus, this shows the following.

Proposition 1.16. *For any spectrum X, the E_2-term of the Adams spectral sequence is given by* $\mathrm{Ext}_A(H^*X, \mathbb{F}_p)$.

1.3. **Convergence of the classical ASS.** We now turn to the issue of convergence of the Adams spectral sequence. We should first explain what this means. Roughly speaking, we say a spectral "converges" when it actually computes what we want it to; or rather we say it converges to G if it allows us to compute G. A lot of the technical material in this subsection is taken directly from [15], but I wholeheartedly recommend the influential paper [5].

So what does that mean? Let's reexamine what we have at hand with the Adams spectral sequence a bit more. Let's fix a spectrum X and let (X_s, g_s) be an Adams resolution of X. Then, in particular, we have a tower of spectra

$$\cdots \to X_s \to X_{s-1} \to \cdots X_1 \to X_0 = X$$

and we have the cofibres of these maps K_s. Now by definition, the E_1-term is the homotopy groups $\pi_* K_s$. To say that the spectral sequence "computes" $\pi_* X$, or at least something related to $\pi_* X$, is to say that what ever survives the spectral sequence actually gives an element of $\pi_* X$ (or something related to $\pi_* X$). We should make some definitions before proceeding.

Definition 1.17. Let $\{E_r, d_r\}$ be a spectral sequence. We say that an element $x \in E_1$ is a *permanent cycle* or an *infinite cycle* if $d_r x = 0$ for all r. We further say that x is a *non-zero permanent cycle* if there is no page E_r and no $y \in E_r$ so that $d_r y = x$.

Remark 1.18. Intuitively, a permanent cycle is just a class $x \in E_1$ which never supports a differential on any subsequent page. Consequently, this yields a class in E_∞. To say that x is a non-trivial permanent cycle simply means that x is never the target of a differential.

Recall that, intuitavely, a class $x \in \pi_* K_s$ survives to E_r if there is a lift $\widetilde{\partial x}$ of ∂x to X_{s+r}, and that $d_r x$ is roughly the projection of $\widetilde{\partial x}$ to $\pi_* K_{s+r}$. Thus, $d_r x = 0$ means that $\widetilde{\partial x}$ can be lifted one further to $\pi_* X_{s+r+1}$. This means, that a permanent cycle x has the property that ∂x lifts to $\pi_* X_{s+r}$ for any r. Thus, ∂x lifts to a class in the homotopy of $\operatorname{holim} X_s$.

Now we want it to be the case that if x is a permanent cycle then we obtain an actual class in $\pi_* X$. Since we have a cofibre sequence

$$X_{s+1} \xrightarrow{g_s} X_s \xrightarrow{f_s} K_s \xrightarrow{\partial} \Sigma X_{s+1}$$

we would be able to lift x to an element $\widetilde{x} \in \pi_* X_s$ provided that $\partial x = 0$. If this is the case, then we get an element of $\pi_* X$ by looking at the image of $\widetilde{x}$ under the map

$$\pi_* X_s \to \pi_* X.$$

Let us summarize the discussion thus far.

(1) We have seen that if $x \in E_1^s = \pi_* K_s$ is a permanent cycle, then we have produced a lift of $\partial x \in \pi_* X_{s+1}$ to the homotopy of $\operatorname{holim} X_s$,
(2) In order get elements of $\pi_* X$, we were naturally led to consider the filtration $F^s \pi_* X$ defined by

$$F^s \pi_u X := \operatorname{im}(\pi_u X_s \to \pi_u X).$$

Note that this is a decreasing filtration of $\pi_* X$.

In light of (1), in order to get actual elements of $\pi_* X$, we would want $\operatorname{holim} X_s \simeq *$. If this is the case then if $x \in E_1$ is a permanent cycle, then ∂x lifts all the way to $\operatorname{holim} X_s$. As $\operatorname{holim} X_s$ is contractible, this implies that $\partial x \sim 0$. So x lifts to class $\widetilde{x}$ in $\pi_* X_s$. We can now state what we mean by convergence.

Definition 1.19. The Adams spectral sequence *converges* if the following two conditions hold:

(1) $\varprojlim_s F^s \pi_u X_s = \bigcap_s F^s \pi_u X_s = 0$, and
(2) There are isomorphisms $E_\infty^{s,t} \cong E_0^s \pi_{t-s} X := F^s \pi_{t-s} X / F^{s+1} \pi_{t-s} X$.

Remark 1.20. The first condition is phrased by saying that the filtration $F^\bullet \pi_u X$ is a *Hausdorff filtration*.

Exercise 3. Show that there is always a map $E_0^s \pi_* X \to E_\infty^{s,*}(X)$ for every spectrum X. This map is always injective.

The second condition in the definition of convergence says that the relationship between the E_∞-term of the spectral sequence and the homotopy groups of $\pi_* X$ is that the former gives an associated graded of the latter. So the E_∞-term is still far from the actual answer we seek. However, there is a way, at least in principal, to reconstruct $\pi_u X$ from the associated graded.

In order to obtain $\pi_u X$ from the associated graded, we may proceed as follows. Note that we have the short exact sequence

$$0 \to F^1\pi_u X \to F^0\pi_u X \to E_\infty^{0,u}(X) \to 0.$$

But this is not the best short exact sequence to think about; rather we should mod out by $F^2\pi_u X$ to obtain the following short exact sequence

$$0 \to F^1\pi_u X/F^2\pi_u X \to F^0\pi_u X/F^2\pi_u X \to E_\infty^{0,u}(X) \to 0.$$

Note that the first term is just $E_\infty^{1,u+1}(X)$. Once we determine $F^0\pi_u X/F^2\pi_u X$ we would like to determine $F^0\pi_u X/F^3\pi_u X$. Note that there is a short exact sequence

$$0 \to F^2\pi_u X/F^3\pi_u X \to F^0\pi_u X/F^3\pi_u X \to F^0\pi_u X/F^2\pi_u X \to 0.$$

Again, note that the first term is the same as $E_\infty^{2,u+2}(X)$.

Continuing in this way, we can in principle reconstruct $\pi_u X/F^s\pi_u X$ for all s. Now, a priori, the filtration may be infinite, and so none of these groups may be $\pi_u X$. But observe that all of these groups fit into a tower,

$$\pi_u X \to \cdots \to \pi_u/F^3\pi_u X \to \pi_u X/F^2\pi_u X \to \pi_u X/F^1\pi_u X.$$

So to obtain $\pi_u X$, it seems reasonable to take the inverse limit $\varprojlim_s \pi_u X/F^s\pi_u X$. There is certainly a map

$$\pi_u X \to \varprojlim_s \pi_u X/F^s\pi_u X.$$

However, this map can fail to be an isomorphism. The following terminology of Boardmann is especially useful for distinguishing between different notions of convergence.

Definition 1.21 (cf. [5])**.** Let $\{E_r, d_r\}$ be a spectral sequence with a filtered target group G. We say that the spectral sequence

(1) *converges weakly* to G if the filtration exhausts G and we have isomorphisms $E_\infty^s \cong F^sG/F^{s+1}G$,
(2) *converges* to G if (1) holds and the filtration is Hausdorff, and
(3) *converges strongly* if (2) and the natural map $G \to \varprojlim_s G/F^sG$ is an isomorphism.

Lemma 1.22. *If X is a spectrum with an Adams resolution (X_s, g_s) such that* $\operatorname{holim} X_s \simeq *$[1]*, then the Adams spectral sequence converges.*

[1]Given a tower of spectra $\cdots X_2 \to X_1 \to X_0$, the homotopy limit can be defined as the fiber of the map

$$\prod_s X_s \to \prod X_s$$

Proof. See Ravenel for the first part, [15].

Now let's identify the E_∞-term. Suppose $[x] \in E_\infty^{s,t}$ is a non-zero class. We first show that $\partial_{s,u}(x) = 0$. Since $d_r[x] = 0$ for all r, the element $\partial_{s,u}(x)$ lifts to $\pi_{u-1}(X_{s+r+1})$ for each r. Thus, $\partial_{s,u}$ is necessarily in the image of $\lim \pi_{u-1}(X_{s+r}) = 0$. So $\partial_{s,u}(x) = 0$. Thus the class x lifts to $\pi_u(X_s)$.

It suffices to show that x has a non-trivial image in $\pi_u(X)$. If not, let r be the largest integer such that the image of x in $\pi_u(X_{s-r+1})$ is nontrivial; let its image be z. Then since the image of z under the map

$$\pi_u X_{s-r} \to \pi_u X_{s-r+1}$$

is 0, it follows that there is a $w \in \pi_{u+1}(K_{s-r})$ such that $\partial_{u+1,s-r}(w) = z$. But this shows that $d_r[w] = x$, contradicting the nontriviality of $[x]$. □

Thus, we are after a general condition on X which guarantees that there is there is an Adams resolution (X_s, g_s) such that $\operatorname{holim}_s X_s \simeq *$.

Lemma 1.23. *Suppose that X is a connective spectrum with each $\pi_i X$ a finite p-group. Then any mod p Adams resolution (X_s, g_s) of X satisfies* $\operatorname{holim}_s X_s$.

Proof. See Ravenel [15]. □

Now we can show that the mod p Adams spectral sequence converges to the p-adic homotopy groups of X. First, I need remind you of some basic facts about Bousfield localization.

Theorem 1.24 (Bousfield)**.** *Let X be a spectrum and let S/p denote the mod p Moore spectrum. If X has finite homotopy groups, then the Bousfield localization $L_{S/p}X$ has homotopy groups given by*

$$\pi_*(L_{S/p}X) \cong \pi_*(X) \otimes \mathbb{Z}_p.$$

Moreover, the canonical map $X \to L_{S/p}X$ induces the obvious map in homotopy groups.

add citation

Remark 1.25. For this reason, people often write $X_p^\wedge$ for $L_{S/p}X$. Also, if $\pi_i X$ is finite for each i, then

$$\pi_i(X) \otimes \mathbb{Z}_p \cong \pi_i(X)[p^\infty],$$

that is $\pi_(X) \otimes \mathbb{Z}_p$ is identified with the p-torsion in $\pi_i(X)$.

proof of 1.5(2). It is required that we identify the E_∞-term of the Adams spectral sequence of X. □

which is $\mathrm{id} - g$. Here g is the map defined so that its component on X_j is given by

$$\prod X_s \xrightarrow{p_{j+1}} X_{j+1} \xrightarrow{f_j} X_j.$$

Example 1.26. Let $X = H\mathbb{Z}$ denote the integral Eilenberg-MacLane spectrum. There is clearly a cofiber sequence

$$X \to X \to H.$$

We then obtain a diagram

$$\begin{array}{ccccccc} X & \longleftarrow & X & \longleftarrow & X & \longleftarrow & \cdots \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\ H & & H & & H & & \end{array}$$

Its easy to check that this is an Adams resolution. In this case, the corresponding Adams spectral sequence has

$$E_1^{s,t} = \begin{cases} \mathbb{Z}/p & s = t \\ 0 & s \neq t. \end{cases}$$

So there is no room for Adams differentials and the spectral sequence collapses at E_1. This shows that we have $E_\infty^{s,s} = \mathbb{Z}/p$. In this case $X_p^\wedge = H\mathbb{Z}_p$.

Exercise 4. Show in this case that $\operatorname{holim}_s X_s$ is contractible after p-completion.

Remark 1.27. The Adams spectral sequence does not converge in general. In particular, it is not typical for it to converge in the case when X is a non-connective spectrum. For example, let $X = KU$ be the complex K-theory spectrum. Then it can be shown that $H_*KU = 0$. Thus, the E_2-term of the ASS for KU is 0. However, Bott showed that $\pi_*KU \cong \mathbb{Z}[\beta^{\pm 1}]$, where $\beta \in \pi_{-2}KU$. So the Adams spectral sequence does not converge in this case.

Exercise 5. The converse if the previous lemma is false: it is not the case that if X is non-connective then it is necessarily the case that the Adams SS fails to converge for X. Provide an example showing this.

Remark 1.28. The index s is often called *Adams filtration*, and if $x \in \pi_*X$, then we say that the *Adams filtration* of x, denoted $AF(x)$, is the least s such that $x \in F^s\pi_*X$.

Now the entire discussion above, we applied the functor $\pi_*(-) = [S^0, -]_*$ to an Adams resolution. We could equally well have considered the functor $[Y, -]_*$ for more general spectra Y. We find the following (see [5] for more details).

Theorem 1.29. *Let X be a spectrum of finite type and Y any spectrum. Then there is a conditionally convergent spectral sequence*

$$E_2^{s,t} = \operatorname{Ext}_A(H^*X, H^Y) \implies [Y, X]_* \otimes \mathbb{Z}_p.$$

If H^*Y is also bounded and finite for each $*$, then the spectral sequence converges strongly.

Convergence of the Adams spectral sequence can be used to show the following theorem of Margolis.

Theorem 1.30 (Margolis)**.** *Let X be any spectrum of finite type. Then the cohomology functor $H^*(-)$ induces an isomorphism*

find citation

$$[H,X]_* \to \mathrm{hom}_A(H^*X,A).$$

Proof. The hypothesis of the previous theorem are met, and so the Adams spectral sequence

$$\mathrm{Ext}_A(H^*X,A) \Longrightarrow [H,X]_* \otimes \mathbb{Z}_p$$

converges. It is a result of Margolis that A is also an injective module over itself, and so the E_2-term of this spectral sequence is concentrated in the line $s=0$. So there is no room for differentials and hence collapses at E_2. This gives the desired isomorphism. □

cite

Exercise 6. Use this to show that DH, the Spanier-Whitehead dual of H, is trivial.

1.4. **Independence of the resolution.** Thus far, it seems that the Adams spectral sequence depends on our choice of Adams resolution. We will sketch an argument showing that, in fact, the spectral sequence is independent of the choice of resolution from the E_2-page onward.

Exercise 7. Show that if $f : X \to Y$ is a morphism of spectra and (X_s, g_s) and (Y_s, h_s) are Adams resolutions of X and Y respectively, then there is a morphism of Adams resolutions $(X_s, g_s) \to (Y_s, h_s)$ which lifts f. You should interpret this in the stable homotopy category of course.

Exercise 8. Deduce from the previous exercise that if f induces an isomorphism in mod p cohomology that the Adams spectral sequences for X and Y are isomorphic on the E_2-term onward (dependent on those chosen resolutions). Using that $X \to L_{S/p}X$ is an isomorphism on mod p homology, show that

2. The generalized Adams spectral sequence

Let E be a ring spectrum. Then, as before, we can consider Adams resolutions based on E. Just replace each instance of H in canonical resolution for X. This gives rise to the *canonical E-based Adams resolution* of X.

$$
\begin{array}{ccccccc}
X & \longleftarrow & \Sigma^{-1}\overline{E}\wedge X & \longleftarrow & \Sigma^{-2}\overline{E}^{\wedge 2}\wedge X & \longleftarrow & \cdots \\
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\
E\wedge X & & \Sigma^{-1}E\wedge\overline{E}\wedge X & & \Sigma^{-2}E\wedge\overline{E}^{\wedge 2}\wedge X & &
\end{array}
.
$$

Applying π_* gives a spectral sequence with E_1-term

$$E_1^{s,t} = \pi_{t-s}\overline{E}^{\wedge s}\wedge X.$$

Unfortunately, we don't automatically obtain an algebraic description of the E_2-term. We need an extra assumption, namely that E_*E is flat over E_*.

Definition 2.1. A *(commutative) Hopf algebroid* over a commutative ring k is a cogroupoid object in the category of (graded, bigraded) commutative k-algebras. That is, its a pair (A,Γ) so that the pair of functors $(\hom_k(A,-),\hom_k(\Gamma,-))$ take values in groupoids.

By the Yoneda lemma this translates into structure maps

$$A \overset{s}{\underset{t}{\rightrightarrows}} \Gamma \overset{\varepsilon}{\longrightarrow} A$$

$$\psi:\Gamma\to\Gamma\otimes_A\Gamma$$

and

$$c:\Gamma\to\Gamma.$$

These maps correspond to the various maps relating morphisms and objects in a groupoid. For example, s and t correspond to the source and target of a morphism, and ε corresponds to map which takes an object to its identity morphism. The morphism ψ arises from composition of composable pairs in the groupoid. The morphism c arises since, in groupoids, each arrow has an inverse.

Remark 2.2. In the literature, the maps s and t are often written as η_L and η_R respectively, and they are called the left and right units.

The fact that (A,Γ) is a cogroupoid implies a number of relations amongst the morphisms above. For example, ψ is a coassociative since composition in groupoids is associative. Another example is that

$$c\circ\eta_L=\eta_R$$

which comes from the fact that if we invert a morphism then we have switched the source and target.

Definition 2.3. something something comodules... go read the goddamn greenbook already! jeez

Theorem 2.4. *If E is a commutative ring spectrum such that E_*E is flat as a module over E_* (via η_L) then E_*E is a Hopf algebroid. Moreover, for any spectrum X, the homology groups E_*X form a left comodule over E_*E.*

Proof. For a proof, see Switzer 17.8. Better yet do it as an exercise! □

The key point in proving the above theorem is the observation that the natural map

$$E_*E \otimes_{E_*} E_*X \to E_*(E \wedge X)$$

is an isomorphism. This is seen by noting that both sides form a homology theory and that this natural transformation is an isomorphism when $X = S^0$.

Theorem 2.5. *If (A, Γ) is a Hopf algebroid such that Γ is flat as an A-module (via η_L), then the category of left (A, Γ)-comodules is an abelian category with enough injectives.*

Proof. See [15, Theorem A1.1.3] and [15, Lemma A1.2.2] □

Theorem 2.6 (Adams)**.** *Let E be a ring spectrum such that E_*E is flat over E_*. An E_*-Adams resolution for X leads to a natural spectral sequence $\{E_r^{**}(X)\}$ with differentials $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ such that*

- $E_2^{s,t} = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X)$
- *the spectral sequence converges to $\pi_*(X_E^\wedge)$ if and only if $\lim^1_r E_r^{s,t} = 0$ for all s, t.*

Proof. See Bousfield's localization paper, section 6. □

Remark 2.7. Very often the E-nilpotent completion $X_E^\wedge$ of X can be identified with the Bousfield localization $L_E X$.

3. Computations with the classical Adams spectral sequence

First we describe how to calculate the Adams E_2-term via the May spectral sequence, which we calculate in a range. We then give an argument showing that $d_2(h_4) = h_0 h_3^2$.

3.1. **Review of the Steenrod algebra.**

Let's recall some facts about the Steenrod algebra. We continue to fix a prime p and let H denote the Eilenberg-MacLane spectrum $H\mathbb{F}_p$.

Theorem 3.1. *The Steenrod algebra $A := [H, H]$ is given by the following:*

(1) *If $p=2$, then A is the free graded associative $\mathbb{F}_2$-algebra generated by the* Steenrod operations Sq^n *in degree n modulo the* Adem relations*: if $a<2b$ then*

$$\mathrm{Sq}^a\mathrm{Sq}^b=\sum_{i=0}^{\lfloor a/2\rfloor}\binom{b-i-1}{a-2i}\mathrm{Sq}^{a+b-i}\mathrm{Sq}^i.$$

(2) *If $p>2$ then A is the free graded associative $\mathbb{F}_p$-algebra generated by the Bockstein β and the Steenrod power operations P^n in degree $2(p-1)n$ modulo the* Adem relations*:For $a<pb$ then*

$$P^aP^b=\sum_i(-1)^{a+i}\binom{(p-1)(b-i)-1}{a-pi}P^{a+b-ii}$$

and for $a\leq pb$

$$P^a\beta P^b=\sum_i(-1)^{a+i}\binom{(p-1)(b-i)}{a-pi}\beta P^{a+b-i}P^i+\sum_i(-1)^{a+i}\binom{(p-1)(b-i)-1}{a-pi-1}P^{a+b-i}\beta P^i.$$

Now the Steenrod algebra is not just an $\mathbb{F}_p$-algebra, it is in fact a *Hopf algebra*.

Definition 3.2. Let k be a commutative ring. A *coalgebra* over k is a k-module Γ together with k-linear maps $\varepsilon:\Gamma\to k$ and $\Delta:\Gamma\to\Gamma\otimes_k\Gamma$, referred to as the *augmentation* and *coproduct* respectively, which are subject to the following conditions:

(1) the coproduct is *counital*, this means the following diagram commutes

$$\begin{array}{ccccc}
\Gamma\cong k\otimes_k\Gamma & \xleftarrow[\varepsilon\otimes\Gamma]{} & \Gamma\otimes_k\Gamma & \longrightarrow & \Gamma\otimes_k k\cong\Gamma\\
 & {\scriptstyle id}\nwarrow & \uparrow{\scriptstyle\Delta} & \nearrow{\scriptstyle id} & \\
 & & \Gamma & &
\end{array}$$

(2) the coproduct is *coassociative*, this means that the following diagram commutes

$$\begin{array}{ccc}
\Gamma & \xrightarrow{\Delta} & \Gamma\otimes\Gamma\\
\downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta\otimes\Gamma}\\
\Gamma\otimes\Gamma & \xrightarrow{\Gamma\otimes\Delta} & \Gamma\otimes\Gamma\otimes\Gamma
\end{array}$$

Definition 3.3. A k-bialgebra is a k-module Γ equipped with the structure of an algebra $\eta:k\to\Gamma$, $\mu:\Gamma\otimes\Gamma\to\Gamma$ and a coalgebra $\varepsilon:\Gamma\to k$, $\Delta:\Gamma\to\Gamma\otimes\Gamma$ which satisfy the following compatibilities,

(1) The following diagram commutes,

$$\begin{array}{ccccc}
\Gamma\otimes\Gamma & \xrightarrow{\mu} & \Gamma & \xrightarrow{\Delta} & \Gamma\otimes\Gamma \\
\big\downarrow{\scriptstyle \Delta\otimes\Delta} & & & & \big\uparrow{\scriptstyle \mu\otimes\mu} \\
\Gamma\otimes\Gamma\otimes\Gamma\otimes\Gamma & & \xrightarrow{\Gamma\otimes\tau\otimes\Gamma} & & \Gamma\otimes\Gamma\otimes\Gamma\otimes\Gamma
\end{array}$$

(2) the following diagram commutes,

$$\begin{array}{ccc}
\Gamma\otimes\Gamma & \xrightarrow{\mu} & \Gamma \\
& \searrow^{\varepsilon\otimes\varepsilon} & \big\downarrow{\scriptstyle \varepsilon} \\
& & k\otimes k\cong k
\end{array}$$

(3) the following diagram commutes

$$\begin{array}{ccc}
k\cong k\otimes k & \xrightarrow{\eta} & \Gamma \\
& \searrow^{\eta\otimes\eta} & \big\downarrow{\scriptstyle \Delta} \\
& & \Gamma\otimes\Gamma
\end{array}$$

(4) the following diagram commutes

$$\begin{array}{ccc}
k & \xrightarrow{\eta} & \Gamma \\
& \searrow^{id} & \big\downarrow{\scriptstyle \varepsilon} \\
& & k
\end{array}$$

Exercise 9. Show that the above axioms imply, for example, that Δ is a morphism of algebras and μ is a map of coalgebras.

Definition 3.4. A *Hopf algebra* is a k-bialgebra Γ with a k-linear map $\chi : \Gamma \to \Gamma$ called the *antipode* or *conjugation map* which makes the following diagram commute

$$\begin{array}{ccccc}
& \Gamma\otimes\Gamma & \xrightarrow{\chi\otimes 1} & \Gamma\otimes\Gamma & \\
{\scriptstyle \Delta}\nearrow & & & & \searrow{\scriptstyle \mu} \\
\Gamma & \xrightarrow{\varepsilon} & k & \xrightarrow{\eta} & \Gamma \\
{\scriptstyle \Delta}\searrow & & & & \nearrow{\scriptstyle \mu} \\
& \Gamma\otimes\Gamma & \xrightarrow{1\otimes\chi} & \Gamma\otimes\Gamma &
\end{array}$$

Remark 3.5. There is an obvious version of a *graded Hopf algebra*. Simply let Γ be a graded k-algebra and require that ε and Δ be grading preserving morphisms.

add specific citation

Wait... is this actually right or are there some extra hypothesis that are required? like that we restrict to commutative k-algebras?

Remark 3.6. In topology, we always work with graded Hopf algebras. Very often, we even work with *connective graded Hopf algebras*; these are graded Hopf algebras Γ such that $\Gamma_n = 0$ for $n < 0$ and $\Gamma_0 = k$. In this situation, we often ignore the antipode χ because it is necessarily uniquely determined from the other structure. See May-Ponto for details

Exercise 10. Another way of defining a Hopf algebra Γ over k is as a *cogroup object* in k-algebras. This means that the functor

$$\hom_k(\Gamma, -) : \mathsf{Alg}_k \to \mathsf{Set}$$

actually lifts to a functor into the category of groups. Using the Yoneda lemma, show that this definition and the one given above are equivalent.

So how, exactly, is the Steenrod algebra a Hopf algebra? The augmentation is given by

$$\varepsilon : A \to \mathbb{F}_2; Sq^i \mapsto 0$$

and

$$\Delta : A \to A \otimes_{\mathbb{F}_2} A; Sq^n \mapsto \sum_{i+j=n} Sq^i \otimes Sq^j.$$

Note that Δ is just given by the Cartan formula and that this makes A into a Hopf algebra.

Exercise 11. Show that this makes A into a cocommutative Hopf algebra.

Now observe that A is locally of finite type. This means that in any given degree, A is finite dimensional. A universal coefficient theorem argument implies that

$$H_*H = \pi_*(H \wedge H) \cong \hom_{\mathbb{F}_p}(A, \mathbb{F}_p) = A_*.$$

The right hand term means we are taking the degree-wise $\mathbb{F}_p$-linear dual.

Exercise 12. Suppose that Γ is a coassociative Hopf algebra over a field k and that Γ is locally of finite type. Let Γ_* denote the degreewise k-linear dual. Show that Γ_* is also a coassociative Hopf algebra. Show further that if Γ is cocommutative, then Γ_* is a commutative algebra.

So by this exercise, the dual of the Steenrod algebra, A_* is a commutative Hopf algebra over $\mathbb{F}_p$. This observation is originally due to Milnor ([12]). He showed that the dual Steenrod algebra has an especially nice algebra structure.

Theorem 3.7 (Milnor, [12])**.** *Let A_* denote the dual Steenrod algebra. Then A_* is a commutative noncocommutative Hopf algebra. Moreover,*

(1) *For* $p = 2$, $A_* = P(\xi_1, \xi_2, \ldots)$ *where* $P()$ *denotes a polynomial algebra over* $\mathbb{F}_p$ *on the indicated generators, and* $|\xi_n| = 2^n - 1$. *The coproduct*

$$\Delta : A_* \to A_* \otimes A_*$$

is given by

$$\Delta\xi_n = \sum_{i+j=n} \xi_j^{2^i} \otimes \xi_i.$$

where $\xi_0 = 1$.

(2) *For* $p > 2$, *then*

$$A_* \cong P(\xi_1, \xi_2, \ldots) \otimes E(\tau_0, \tau_1, \ldots)$$

where $E()$ *denotes an exterior algebra over* $\mathbb{F}_p$ *on the indicated generators, and* $|\xi_n| = 2(p^n - 1)$ *and* $|\tau_n| = 2p^n - 1$. *The coproduct is given by*

$$\Delta\xi_n = \sum_{i+j=n} \xi_j^{p^i} \otimes \xi_j$$

and

$$\Delta\tau_n = \tau_n \otimes 1 + \sum_{i+j=n} \xi_j^{p^i} \otimes \tau_i$$

(3) *For all* p, *the conjugation map* $\chi : A_* \to A_*$ *is given by*

$$\chi(1) = 1$$

and

$$\sum_{i+j=n} \xi_i^{p^j} \chi\xi_j = 0$$

and

$$\tau_n + \sum_{i+j}^{n} \xi_i^{p^j} \chi\tau_j = 0$$

Now, in order to calculate with the Adams spectral sequence, we better know how to calculate the E_2-term. Given Milnor's theorem, we often prefer to calculate with *homology* and the *dual Steenrod algebra*. So we want to dualize. This comes at a cost; we have to think of things in terms of comodules rather than modules.

Definition 3.8. Let Γ be a Hopf algebra. Then a *left comodule* C over Γ is a k-module equipped with a k-linear map

$$\alpha : C \to \Gamma \otimes C$$

which makes the following diagrams commute

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & \Gamma \otimes C \\ \downarrow & & \downarrow{\scriptstyle \Gamma\otimes\alpha} \\ \Gamma \otimes C & \xrightarrow{\Delta \otimes C} & C \otimes \Gamma \otimes \Gamma \end{array}$$

and

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & \Gamma \otimes C \\ & {\scriptstyle id}\searrow & \downarrow{\scriptstyle \varepsilon\otimes C} \\ & & k \otimes C \cong C \end{array}$$

Exercise 13. If M is an A-module which is locally finite, show that the degreewise $\mathbb{F}_p$-linear dual M^* is an A_*-comodule. Conclude that for reasonable spectra X that H_*X is an A_*-comodule.

Now I have made the case that we want to use the dual Steenrod algebra A_* for computations, and I have indicated that we must work with comodules rather than modules. But in order to make this viable for actually computing anything, we need to be able to express the Adams E_2-term in terms of comodules. The nicest relationship that you could hope for is

$$\mathrm{Ext}_A(M, \mathbb{F}_p) \cong \mathrm{Ext}_{A_*}(\mathbb{F}_p, M^*)$$

or more generally

$$\mathrm{Ext}_A(M, N) \cong \mathrm{Ext}_{A_*}(N^*, M^*).$$

Of course, this would require us to make sense of what Ext of comodules means. In order to do that, we would have to argue that the category of comodules is abelian with enough injectives or something like that. For the time being let's put these foundational issues aside and and just take for granted that this can be done.

We should also explain a bit more how the Hopf algebra structure on A_* arises topologically. Recall that $A_* = \pi_* H \wedge H$. The coproduct for A_* is given by

$$H \wedge H \simeq H \wedge S^0 \wedge H \to H \wedge H \wedge H.$$

The homotopy groups of the target of this map is

$$H_*(H \wedge H) \cong H_*H \otimes H_*H \cong A_* \otimes A_*$$

by the Künneth theorem. If X is any spectrum, then we get that H_*X is a comodule over A_* via the following map

$$H \wedge X \simeq H \wedge S^0 \wedge X \to H \wedge H \wedge X.$$

Again the target of this morphism has, by the Künneth theorem, homotopy groups given by the following

$$H_*(H \wedge X) \cong H_*H \otimes H_*X \cong A_* \otimes H_*X.$$

Finally, the conjugation χ on A_* is induced by the switching morphism

$$\tau : H \wedge H \to H \wedge H$$

In order to do some computations later, we need an expicit complex that calculates $\mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*X)$. This is given by the so-called *cobar complex*.

Definition 3.9. Let $\overline{A}_*$ be $\ker \varepsilon$.

Definition 3.10. Let C be a comodule over A_*. The *cobar complex* $C^\bullet_{A_*}(C)$ is defined by

$$C^s_{A_*}(C) := \overline{A}_*^{\otimes s} \otimes C.$$

We denote elements $a_1 \otimes \cdots \otimes a_s \otimes x$ by $[a_1|\ldots|a_s]x$. The differential d : $C^s_{A_*}(C) \to C^{s+1}_{A_*}(C)$ is defined by

$$d[a_1 \mid \cdots \mid a_s]x = [1 \mid a_1 \mid \cdots \mid a_s]x + \sum_{i=1}^{s}(-1)^i[a_1 \mid \cdots \mid a_{i-1} \mid a_i' \mid a_i'' \mid a_{i+1} \mid \cdots \mid a_s]x + (-1)^{s+1}[a_1 \mid \cdots \mid a_s$$

where we have $\Delta a_i = a_i' \otimes a_i''$ and $\alpha(x) = x' \otimes x'' \in A_* \otimes C$.

Finally, before moving on, I want to state a useful theorem for computations.

Theorem 3.11 (Change-of-Rings Isomorphism)**.** *Let A be an algebra and $B \subseteq A$ a subalgebra such that A is flat over B as a right B-module. Let M be a left B-module and let N be a left A-module. Then there is a natural isomorphism*

$$\mathrm{Ext}^{s,t}_A(A \otimes_B M, N) \cong \mathrm{Ext}^{s,t}_B(M, N).$$

Proof. Let $P_* \to M$ be a B-free resolution. Then $A \otimes_B P_* \to A \otimes_B M$ is an A-free resolution of $A \otimes_B M$. The result follows because of the isomorphism

$$\hom_A(A \otimes_B P_*, N) \cong \hom_B(P_*, N).$$

□

Definition 3.12. If A is an augmented k-algebra and $B \subseteq A$ a subalgebra, then we shall often write $A /\!\!/ B$ for $A \otimes_B k$.

Using this notation, we then have

$$\mathrm{Ext}_A(A /\!\!/ B, N) \cong \mathrm{Ext}_B(k, N).$$

Now suppose that A is connected graded Hopf algebra over a field k which is locally finite, so that A_* is also a Hopf algebra. If M and N are two A-modules which are locally finite, then M^* and N^* are comodules. If B is a sub-Hopf algebra of A, then the tensor product $M \otimes_B N$ can be described as the coequalizer

$$M \otimes B \otimes N \rightrightarrows M \otimes N \longrightarrow M \otimes_B N.$$

Dualizing leads us to the *cotensor product*.

Definition 3.13. Let B be a coalgebra. The *cotensor product* of a right B-comodule M with a left B-comodule N is the equalizer

$$M \,\square_B\, N \longrightarrow M \otimes N \underset{M \otimes \alpha_N}{\overset{\alpha_M \otimes N}{\rightrightarrows}} M \otimes B \otimes N.$$

Thus, in the context of a Hopf algebra we can dualize to get a new version of the change of rings theorem.

Theorem 3.14. *Let Γ be a Hopf algebra over a field k and Σ a Hopf algebra quotient of Γ, in other words there is a surjective morphism $A \to B$ of Hopf algebras. If N is a left comodule over Σ, then*

$$\mathrm{Ext}_A(k, \Gamma \,\square_\Sigma\, N) \cong \mathrm{Ext}_\Sigma(k, N).$$

In the case $N = k$ above, then we write $\Gamma \,\square_\Sigma\, k$ as $\Gamma /\!/ \Sigma$. Here, we are regarding k as a Σ-comodule via the unit map

$$\eta : k \to \Sigma.$$

Exercise 14. Show, from the definition, that if Γ is a Hopf algebra over k and M is a right Γ-comodule, then

$$M \,\square_\Gamma\, k = \{x \in M \mid \alpha(x) = x \otimes 1\}.$$

Finally, I want to tell you about some important subalgebras of A as well as some important elements in A.

Definition 3.15. Set $Q_0 := \beta$ to be the Bockstein element (when $p = 2$ this should be interpreted as Sq^1). Inductively define elements by

$$Q_k := [P^{p^k}, Q_{k-1}].$$

Example 3.16. Let $p = 2$. Then $Q_1 = [\mathrm{Sq}^2, \mathrm{Sq}^1]$, which is exactly the element

$$Q_1 = \mathrm{Sq}^2\mathrm{Sq}_1 + \mathrm{Sq}^3.$$

We also have

$$Q_2 = [\mathrm{Sq}^4, Q_1] = \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^7 + \mathrm{Sq}^5\mathrm{Sq}^2 + \mathrm{Sq}^6\mathrm{Sq}^1$$

Theorem 3.17 (Milnor, [12])**.** *The elements $Q_0^{\varepsilon_0} Q_1^{\varepsilon_1} \cdots P^R$ form an additive basis of A which, up to a sign, dual to the obvious basis for A_*. In particular $Q_i^2 = 0$ and*

$$Q_i Q_j + Q_j Q_i = 0.$$

Also, the elements Q_i are primitive, this means that

$$\Delta(Q_i) = Q_i \otimes 1 + 1 \otimes Q_i.$$

The up shot of this is that we can make the following definitions.

Proposition 3.18. *Define $E := E(Q_0, Q_1, Q_2, \ldots)$. Then E is a subHopf algebra of A. The same is true for $E(n) := E(Q_0, Q_1, \ldots, Q_n)$.*

Definition 3.19. Define $A(n) := \langle \beta, P^1, \ldots, P^{p^n} \rangle$, the subalgebra of A generated by the elements $\beta, P^1, \ldots, P^{p^n}$.

Exercise 15. Show that this defines a subHopf algebra.

Theorem 3.20 (Milnor, [12])**.** *The algebras $A(n)$ are finite dimensional and $A = \operatorname{colim}_n A(n)$. Thus every element of A in positive degree is nilpotent.*

Question 3.21. Is there a nice numerical function which describes, or at least bounds, the order of the P^n? This question is open as far as I know.

Since the $A(n)$ are subalgebras of A, when we dualize we get that $A(n)_*$ is a quotient of A_*.

Theorem 3.22. *As an algebra, for $p = 2$, we have*

$$A(n)_* = A_* / (\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \ldots, \zeta_n^{2^2}, \zeta_{n+1}^2, \zeta_{n+2}, \ldots)$$

3.2. **Some computations.** So let's calculate $\pi_* X$ for various spectra X.

Lets start by using the Adams spectral sequence to calculate $H\mathbb{Z}$.

Theorem 3.23. *There is an isomorphism*

$$H^* H\mathbb{Z} \cong A /\!\!/ A(0) := A \otimes_{A(0)} \mathbb{F}_p$$

where $A(0)$ is the subalgebra of A generated by the Bockstein element.

Note that $A(0)$ is an exterior algebra $E(\beta)$ where β is the mod p Bockstein. It can be shown that the dual $A(0)_*$ is a quotient of A_* and it is given by

$$A(0)_* = \begin{cases} E(\xi_1) & p = 2 \\ E(\tau_0) & p > 2. \end{cases}$$

Exercise 16. Let $E(x)$ be an exterior algebra on a generator x in positive degree over a field k. Show that

$$\operatorname{Ext}_{E(x)}(k, k) \cong k[y]$$

where $|y| = (1, |x|)$. (Hint: For this exercise, its useful to use a minimal resolution to prove one part, and the cobar complex to prove the other.)

Thus, the E_2-term of the Adams spectral sequence for $H\mathbb{Z}$ is given by

$$\mathrm{Ext}_A(A/\!\!/A(0), \mathbb{F}_p) \cong \mathrm{Ext}_{A(0)}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p[h_0]$$

where h_0 is in bi-degree $(1,1)$. Thus the Adams spectral sequence collapses at the E_2-term. Now, we picture this as ...

put picture here.

Now because of the multiplicative structure, this shows that the extensions can't be trivial. Thus we have

$$\pi_*(H\mathbb{Z}_p^\wedge) \cong \mathbb{Z}_p.$$

Theorem 3.24 (Milnor-Moore, Theorem 4.7 of [11])**.** *If A is a connected Hopf algebra over a field k, B a connected left A-comodule over A, $C = k\,\square_A B$ and the maps $B \to A$ and $C \to B$ are surjective and injective respectively, then there exist a morphism $h : A \otimes C \to B$ which is simultaneously an isomorphism of left A-comodules and right C-modules.*

Add citation

Theorem 3.25 (Thom)**.** *The homology of the Thom spectrum MO is given by $H(MO;\mathbb{F}_2) \cong A_* \otimes N$ where N is a polynomial algebra with one generator in each degree not of the form $2^k - 1$ and is a trivial comodule. If p is odd, then $H_*(MO;\mathbb{F}_p) = 0$.*

Proof. I guess you can also prove this using Milnor-Moore. All you need to know (which is proved by Thom), is that the Thom class $u : \mathrm{MO} \to H$ is a surjection in homology. Milnor-Moore then implies that there is an isomorphism of A_*-comodules

$$A_* \otimes N \cong H_* MO$$

where $N = \mathbb{F}_2 \,\square_{A_*} H_* MO$. A degree counting argument then shows that N has the claimed form. $\square$

We can input this into the Adams spectral sequence. We obtain

$$\mathrm{Ext}_{A_*}(H_* MO) \implies \pi_* MO.$$

By the proposition, we can rewrite the E_2-term as

$$\mathrm{Ext}_{A_*}(H_* MO) \cong \mathrm{Ext}_{A_*}(A_* \otimes N) \cong \mathrm{Ext}_{A_*}(A_*) \otimes N \cong N.$$

Note that N is concentrated in Adams filtration 0. So there is no room for differentials or hidden extensions. Thus, we find that

$$\pi_* MO \cong N.$$

Now let's calculate the homotopy groups of MU. We need a few preliminaries. Recall that we defined the Milnor primitives in the last section. Then we have

$$P_* := A /\!\!/ E_* = A_* \square_{E_*} \mathbb{F}_p = \begin{cases} P(\xi_1^2, \xi_2^2, \xi_3^2, \ldots) & p = 2 \\ P(\xi_1, \xi_2, \xi_3, \ldots) & p > 2 \end{cases}$$

Observe the following.

Lemma 3.26. *The subalgebra P_* is a subHopf algebra of A_*.*

Exercise 17. Prove this.

Lemma 3.27. *Suppose that C is a comodule over A_* which is concentrated in even degrees. Then C is naturally a comodule over P_*.*

Proof. Let $x \in C_*$ be a homogenous element. Then

$$\alpha(x) = \sum_i x_i' \otimes x_i'' \in A_* \otimes C.$$

Since x_i'' is in an even degree and since α is a degree preserving, it follows that x_i' is necessarily in even degree. For the sake of concreteness, suppose that $p > 2$. Suppose there were an element x_i' which is not in P_*. Then the only way it could be in even degrees is if x_i' is of the form $m\tau$ where $m \in P_*$ and $\tau \in E_*$ is a product of an even number of distinct τ_i's. On the other hand, by coassociativity we must have that $\Delta(m\tau)$ is entirely in even degrees, but we see from the formulas for $\Delta(\tau)$ that this is impossible if $\tau \neq 0$. □

Theorem 3.28. *We have the following.*

1. $H^*(BU; \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \ldots]$ *where* $|c_i| = 2i$.
2. *The Thom spectrum* MU *then has a Thom class* $u : \mathrm{MU} \to H\mathbb{F}_p$ *and we have* $H_*(\mathrm{MU}; \mathbb{Z}) \cong \mathbb{Z}[b_1, b_2, \ldots]$ *where* $|b_i| = 2i$.
3. *For any prime p, the image of the Thom class u in homology is P_*.*

We will need the following theorem.

Proposition 3.29 (Milnor, Novikov, [10])**.** *The mod p homology of* MU *is given by $P_* \otimes N$ where N is a polynomial algebra on generators x_i of degree $2i$ where $i \neq p^j - 1$. Here P_* is the even subalgebra of A_*.*

Proof. As MU is a ring spectrum, the homology $H_*\mathrm{MU}$ is a comodule algebra over A_*. By the previous theorem, it must then be that $H_*\mathrm{MU}$ is a comodule algebra over P_*. The Thom class gives a surjective map of comodule algebras

$$H_*\mathrm{MU} \to P_*.$$

We take

$$C = \mathbb{F}_p \square_{P_*} H_*\mathrm{MU} = \{x \in H_*\mathrm{MU} \mid \alpha(x) = 1 \otimes x\}$$

By the Milnor-Moore theorem we have an isomorphism of P_*-comodules $H_*\mathrm{MU} \cong P_* \otimes C$. Now, by construction, C has a trivial coaction and is a subalgebra of $H_*\mathrm{MU}$. To show that it has the desired form, one just counts the dimensions of C in any given degree. □

Remark 3.30. Its interesting to read the original papers, especially Milnor's ([10]) which precedes the paper of Milnor-Moore [11]. In [10], Milnor gives a very computational and explicit description of the action of A on $H^*\mathrm{MU}$ to show that C is a trivial comodule which is worth reading.

Plugging this into the Adams spectral gives

$$E_2^{s,t} \cong \mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*\mathrm{MU}) \implies \pi_*\mathrm{MU}.$$

We need to calculate the E_2-term. To do that, we would like to use a change-of-rings isomorphism. We have established that $H_*\mathrm{MU} \cong P_* \otimes C$, and so we have

$$H_*\mathrm{MU} \cong P_* \otimes C \cong (A_* \square_{E_*} \mathbb{F}_p) \otimes_{\mathbb{F}_p} C \cong A_* \square_{E_*} C,$$

where the last isomorphism follows because C is has trivial coaction. We can use the change-of-rings isomorphism to write this as

$$\mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*\mathrm{MU}) \cong \mathrm{Ext}_E(\mathbb{F}_p, C) \cong \mathrm{Ext}_E(\mathbb{F}_p, \mathbb{F}_p) \otimes C.$$

Using the above exercise and the Künneth isomorphism shows that

$$\mathrm{Ext}_E(\mathbb{F}_p) \cong P(v_0, v_1, \ldots),$$

where $|v_i| = (1, 2p^i - 1)$. Note that the E_2-term of this ASS is concentrated in only even total degrees, and so there is no room for differentials. There is also no room for any hidden extensions. This shows that

$$\pi_* MU_p^\wedge \cong \mathbb{Z}_p[v_1, v_2, \ldots] \otimes C.$$

This is just giving the local structure of $\pi_* MU$, but we want the integral structure.

Definition 3.31. Let R be a connected graded ring with $\overline{R}$ the ideal of all elements in positive degree. Then the module of *indecomposables* of R is defined to be $QR := \overline{R}/\overline{R}^2$. To indicate the part of QR in degree i, we write $Q_i R$.

Note that $Q_{2i}\pi_* MU \otimes \mathbb{Z}/p$ is just $\mathbb{Z}/p$ for each $i > 0$. Thus $Q_{2i}\pi_* MU \cong \mathbb{Z}$ for each $i > 0$. Pick an element $x_i \in \pi_{2i}\mathrm{MU}$ which projects to a generator of $Q_{2i}R$ and define

$$L := \mathbb{Z}[x_1, x_2, \ldots].$$

Then there is an obvious map $L \to \pi_* MU$. By the previous computations with the ASS, this map is an isomorphism after tensoring with $\mathbb{Z}_{(p)}$, and hence is an isomorphism globally.

Now let's calculate the homotopy groups of ku using the Adams spectral sequence.

Theorem 3.32 (Adams)**.** *The mod p homology of ku is given by* put in citation

$$H_* ku \cong A /\!\!/ E(Q_0, Q_1)_* \cdot \{1, \beta, \ldots, \beta^{p-2}\}.$$

Plugging this into the Adams spectral sequence for any given prime p, we find that

$$E_2^{**} = \mathrm{Ext}_{A_*}(H_* ku) \cong \mathrm{Ext}_{E(2)_*}(\mathbb{F}_p, \mathbb{F}_p) \otimes \mathbb{F}_p\{1, \beta, \ldots, \beta^{p-2}\}.$$

By a previous exercise we have put in hyper-link to exercise

$$\mathrm{Ext}_{E(2)_*}(\mathbb{F}_p) \cong P(h_0, v_1),$$

where $|v_1| = (1, 2p-1)$. Thus v_1 is in stem $2(p-1)$. So we have that the Adams E_2-term is

$$P(h_0, v_1) \otimes \mathbb{F}_p\{1, \beta, \ldots, \beta^{p-2}\}.$$

Since all the classes are in even total degree, there is no room for differentials. There also no additive extension problems. Similar to the case of MU we have

$$\pi_* ku = \begin{cases} \mathbb{Z} & * \equiv 0 \mod 2 \\ 0 & * \equiv 1 \mod 2 \end{cases}$$

Now, ku is actually a ring spectrum and as an algebra

$$H_* ku \cong A /\!\!/ E(2)_* \otimes \mathbb{F}_p[\beta]/\beta^{p-1}.$$

Thus there is a hidden *multiplicative* extension

$$\beta \cdot \beta^{p-2} = v_1$$

in the Adams spectral sequence.

Remark 3.33. There are various ways to figure out this hidden extension. One way relies on BP, the Brown-Peterson spectrum, and some facts about formal group laws. The theory ku is complex-orientable and carries the multiplicative formal group law. This guarantees a map $\mathrm{BP}_* \to ku_*$, and calculations with formal group laws shows that $v_1 \mapsto \beta^{p-1}$. Really you probably want to start with MU $\to ku$ and then go to BP so that we can use the fact that MU is free.

finish this

3.3. **The May spectral sequence.** In this section I want to give an explanation of a tool that was developed by Peter May to calculate that Adams E_2-term. However, the approach I take follows [15]; May's original approach is actually general enough to give a spectral sequence for computing the cohomology of the universal enveloping algebra of a restricted Lie algebra.

put citation

We will obtain the May spectral sequence by putting a filtration on the dual Steenrod algebra, this will induce a filtration on the cobar complex for A_*, which then yields a spectral sequence. For concreteness, I will fix p to be 2, and then indicate the necessary changes to get the spectral sequence at odd primes.

Definition 3.34. We define a function, referred to as *May weight*, on monomials of A_* as follows: we set $\mathrm{wt}(\xi_i^{2^j}) = 2i-1$ and we extend it to general monomials by taking the dyadic expansion in the powers and then extending multiplicatively. So, for example, we consider ξ_1^7 and rewrite it as

$$\xi_1^7 = \xi_1^{1+2+4} = \xi_1^1\xi_1^2\xi_1^4,$$

then the May weight of ξ_1^7 is

$$\mathrm{wt}(\xi_1^7) = \mathrm{wt}(\xi_1^1\xi_1^2\xi_1^4) = \mathrm{wt}(\xi_1) + \mathrm{wt}(\xi_1^2) + \mathrm{wt}(\xi_1^4) = 1+1+1 = 3.$$

Exercise 18. Calculate the May weight of the following monomials: $\xi_1^5\xi_3^2, \xi_7^3\xi_{10}$, and however many more until you feel like you get it.

We have now defined the May weight for arbitrary monomials in A_*. We can use this to define a filtration of A_*.

Definition 3.35. Define an increasing filtration $F_\bullet A_*$ on A_* by setting F_iA_* to be the subspace of A_* spanned by all monomials of May weight $\le i$.

Exercise 19. Show that this is a multiplicative filtration on A_*.

It is important to note that the coproduct does not increase the May weight. In fact, we can see what happens by explicit computation. The coproduct on ξ_n is given by

$$\psi(\xi_n) = \sum_{i+j=n} \xi_i^{2^j} \otimes \xi_j.$$

Because we are working over $\mathbb{F}_2$, we obtain

$$\psi(\xi_n^{2^k}) = \sum_{i+j=n} \xi_i^{2^{j+k}} \otimes \xi_j^{2^k}.$$

Now $\mathrm{wt}(\xi_n^{2^k}) = 2n-1$. On the other hand, any term in the sum has May weight given by

$$\mathrm{wt}(\xi_i^{2^{j+k}} \otimes \xi_j^{2^k}) = 2i-1+2j-1 = 2n-2.$$

So the coproduct actually decreases the May weight. In particular, this means that the associated graded $E_*^0 A_*$ inherits the structure of a (bigraded) Hopf algebra. In fact, $E_*^0 A_*$ is an especially nice Hopf algebra.

Proposition 3.36. *The Hopf algebra $E_*^0 A_*$ is an exterior Hopf algebra on primitive generators $\xi_{i,j}$, where $\xi_{i,j}$ is represented by $\xi_i^{2^j}$.*

Proof. It is clear that $E_*^0 A_*$ is generated as an algebra by the elements $\xi_{i,j}$. Observe that $\xi_{i,j} \in E^0_{2i-1} A_{2^j-1}$. To check that these are exterior elements, note that, by definition of the associated graded, that

$$\xi_{i,j}^2 \in E^0_{4i-2} A_{2^{j+1}-2}.$$

However, $\xi_{i,j}^2$ is represented by $\xi_i^{2^{j+1}}$, which is in filtration $2i-1$. Thus $\xi_{i,j}^2 = 0$.

It remains to compute the coproduct on $\xi_{i,j}$. Note that the coproduct on $\xi_i^{2^j}$ is

$$\xi_i^{2^j} \otimes 1 + \sum_{n=1}^{i-1} \xi_n^{2^{i-n+j}} \otimes \xi_{n-i}^{2^j} + 1 \otimes \xi_i^{2^j}.$$

As we noticed before, all of the terms in the middle actually have May weight $2i-2$, and are thus 0 in the associated graded. This shows that

$$\psi(\xi_{i,j}) = \xi_{i,j} \otimes 1 + 1 \otimes \xi_{i,j}.$$

□

This filtration of A_* induces a filtration on the cobar complex of $\mathbb{F}_2$. This is defined in the following way

$$F_k C^s_{A_*}(\mathbb{F}_2) := \sum_{i_1+\cdots+i_s \leq k} F_{i_1} A_* \otimes \cdots \otimes F_{i_s} A_*$$

Exercise 20. Show that with the natural filtration on $A_* \otimes A_*$, we have a bigraded isomorphism

$$E_*^0(A_* \otimes A_*) \cong E_*^0 A_* \otimes E_*^0 A_*.$$

By induction we thus have

$$E_*^0(A_*^{\otimes s}) \cong E_*^0(A_*)^{\otimes s}.$$

Since we have a filtration on the cobar complex $C^\bullet_{A_*}(\mathbb{F}_2)$ we get an associated spectral sequence

$$E_1 = H^*(E^0 C^\bullet_{A_*}(\mathbb{F}_2)) \implies H^*(C^\bullet_{A_*}(\mathbb{F}_2)).$$

But in fact we have

$$E^0_* C^\bullet_{A_*}(\mathbb{F}_2) \cong C^\bullet_{E^0 A_*}(\mathbb{F}_2)$$

which means we can rewrite this spectral sequence using slightly more familiar terms.

$$\mathrm{Ext}_{E^0 A_*}(\mathbb{F}_2, \mathbb{F}_2) \implies \mathrm{Ext}_{A_*}(\mathbb{F}_2, \mathbb{F}_2).$$

Since $C^\bullet_{A_*}(\mathbb{F}_2)$ is a cochain complex in graded objects, we have that $C^\bullet_{E^0 A_*}(\mathbb{F}_2)$ is a cochain complex in *bigraded objects*. Thus the E_1-term is a trigraded object.

Now when one writes down a spectral sequence, one should take care to carefully track the indices. The E_1-term for the May spectral sequence is usually indexed as

$$E_1^{s,t,m} = \mathrm{Ext}^{s,m,t}_{E^0 A_*}(\mathbb{F}_2, \mathbb{F}_2),$$

where s denotes the *cohomological degree* and (t, m) denotes the the internal bidegree. This means that $a \in \mathrm{Ext}^{s,t,m}_{E^0 A_*}(\mathbb{F}_2, \mathbb{F}_2)$ is represented by a cocycle $\alpha \in C^s_{E^0 A_*}(\mathbb{F}_2)$ of bidegree (t, m). We refer to m as the May filtration and t as the topological degree.

Exercise 21. Show that the term $E_1^{s,t,m}$ contributes to $\mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_2, \mathbb{F}_2)$.

Thus, people will typically write this spectral sequence as

$$E_1^{s,t,m} = \mathrm{Ext}^{s,t,m}_{E^0 A_*}(\mathbb{F}_2, \mathbb{F}_2) \implies \mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_2, \mathbb{F}_2).$$

In light of Proposition 3.36, we can write the E_1-term as

$$\mathrm{Ext}^{***}_{E^0 A_*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(h_{i,j} \mid i \geq 1, j \geq 0)$$

with $h_{i,j}$ in tri-degree $(1, 2^j(2^i - 1), 2i - 1)$.

Next, let's figure out the direction of the differentials. Recall that a d_r-differential is going to be changing the filtration degree by r. In this case, we obtained our spectral sequence from an *increasing filtration*, so that means that the differential is supposed to *lower* the filtration. Thus, the differentials are supposed have the tri-digree

$$d_r : E_r^{s,t,m} \to E_r^{s+1,t,m-r}.$$

We should also determine, at least, the d_1-differential. Recall we have

$$\psi(\xi_i^{2^j}) = \xi_i^{2^j} \otimes 1 + \sum_{k=1}^{i-1} \xi_{i-k}^{2^{k+j}} \otimes \xi_k^{2^j} + 1 \otimes \xi_i^{2^j}$$

and recall that we noticed that the terms in them middle have May filtration $2i-2$. In particular, they are exactly one lower in May filtration. This shows that

$$d_1(h_{i,j}) = \sum_{k=1}^{i-1} h_{i-k,k+j} h_{k,j}.$$

We have thus shown

Theorem 3.37. *There is a convergent multiplicative spectral sequence of the form*

$$E_1^{s,t,m} \Longrightarrow \mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$$

with

- $E_1 \cong P(h_{i,j} \mid i \geq 1, j \geq 0)$ *with* $|h_{i,j}| = (1, 2^j(2^i-1), 2i-1)$ *and*
- $d_r : E_r^{s,t,m} \to E_r^{s+1,t,m-r}$.
- *The* d_1*-differential is determined by the formula* $d_1(h_{i,j}) = \sum_{k=1}^{i-1} h_{i-k,k+j} h_{k,j}$.

Proof. The only thing we didn't prove in the above discussion is that the spectral sequence is multiplicative. This follows from the standard fact that a multiplicative filtration on a DGA results in a multiplicative spectral sequence. □

Remark 3.38. It is common for people to abbreviate $h_{1,j}$ as h_j. These are the only elements on the $s=1$ line which survive to the E_∞-term of the MSS, and so are the only elements on the 1-line of the Adams E_2-term. These elements are called the *Hopf invariant classes* because if they survive the Adams spectral sequence, then that gives the existence of a Hopf invariant 1 element in that stem.

Remark 3.39. Since $E_\infty^{s,t,m}$ contributes to $\mathrm{Ext}_{A_*}^{s,t}$, we typically surpress m from the notation and draw the May spectral sequence in Adams coordinates $(t-s, s)$. Under this convention, May differentials look like Adams d_1-differentials.

Remark 3.40. We can clearly define analogous filtrations for the quotient Hopf algebras $A(n)_*$. This results in a May spectral sequence for each $A(n)_*$.

Exercise 22. Defining the analogous filtrations on $A(n)_*$ show that the E_1-page is a again a polynomial algebra on some $h_{i,j}$. For which i, j is $h_{i,j}$ in the E_1-page for the May SS for $A(n)_*$?

So let's see how the May spectral sequence works for $A(1)_*$. Recall that $H^*(ko) = A/\!\!/A(1)$, thus the Adams spectral sequence for ko takes the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{A(1)_*}(\mathbb{F}_2) \implies \pi_* ko_2^\wedge.$$

Thus the May SS for $A(1)_*$ calculates the Adams E_2-term for ko. Arguing analogously as above, we find that the May E_1-term for $A(1)_*$ is

$$E_1^{***} = P(h_0, h_1, h_{2,0}),$$

and the d_1-differential is given by

$$d_1(h_{2,0}) = h_0 h_1.$$

This shows that the E_2-term of the spectral sequence is given by

$$P(h_0, h_1, b_{20})/(h_0 h_1)$$

where $b_{20} := h_{20}^2$. Observe that the tri-degree of b_{20} is $(2,6,6)$. We will show that there is only one further May differential. Now, b_{20} survives to the E_2-term because d_1 satisfies the Leibniz rule. So in particular,

$$d_1(h_{20}^2) = d_1(h_{20}h_{20}) = h_0 h_1 h_{20} + h_{20} h_0 h_1 = 0.$$

However, the only reason this is 0 on the E_1-term of the MSS is because the elements of the E_1-term commute with each other.

Let's examine this a bit more closely. In $C^\bullet_{E^0 A(1)_*}(\mathbb{F}_2)$, the element b_{20} is represented by $x = [\xi_{20} \mid \xi_{20}]$. This naturally lifts to $C^\bullet_{E^0 A(1)_*}(\mathbb{F}_2)$ via $\widetilde{x} = [\xi_2 \mid \xi_2]$. We can calculate the cobar differential on $\widetilde{x}$, it is

$$d(\xi_2 \mid \xi_2) = \xi_1^2 \mid \xi_1 \mid \xi_2 + \xi_2 \mid \xi_1^2 \mid \xi_1.$$

This corresponds on the May E_1-page to

$$h_1 h_0 h_{20} + h_{20} h_1 h_0.$$

Of course this element is 0 since the E_1-page is commutative, so it looks like b_{20} might not support a May differential. But wait! We should remember *why* the elements h_1, h_0, h_{20} commute with each other.

Remember that $E_1 = H^*(C^\bullet_{E^0 A(1)_*}(\mathbb{F}_2))$. In this case, we have that $E^0 A(1)_*$ is a primitively generated exterior Hopf algebra

$$E^0 A(1)_* = E(\xi_{1,0}, \xi_{1,1}, \xi_{2,0}).$$

Let us analyze this is in slightly greater generality.

Example 3.41. Let $E = E(x, y)$ be a primitively generated exterior Hopf algebra. Then we know $\mathrm{Ext}_E(\mathbb{F}_2)$ is a polynomial generator on classes h_x and h_y which are represented in the cobar complex by $[x]$ and $[y]$ respectively. The classes $h_x h_y$ and $h_y h_x$ are represented by $[x \mid y]$ and $[y \mid x]$

respectively. In order to show that these elements represent the same class in cohomology , we need to see that

$$x \mid y + y \mid x$$

is a coboundary in the cobar complex. Observe that

$$\psi(xy) = \psi(x)\psi(y) = (x\otimes 1 + 1\otimes x)\cdot(y\otimes 1 + 1\otimes y) = xy\otimes 1 + x\otimes y + y\otimes x + 1\otimes xy.$$

This shows that in the cobar compelx

$$d([xy]) = [x \mid y] + [y \mid x].$$

This suggests how we might find differentials in the May spectral sequence! The only reason it looks like the elements h_0, h_1, h_{20} look like they commute on the level of the May E_1-term is because we *ignored* terms of lower May filtration when computing the diagonals of elements in $E^0A(1)_*$. Thus we should *remember* why h_0, h_1, h_{20} commute in $C^\bullet_{E^0A(1)_*}(\mathbb{F}_2)$ and then lift to $C^\bullet_{A(1)_*}(\mathbb{F}_2)$ and then compute the cobar differential. The residual stuff will lead to a May differential.

So let's see how this works out in this case. As we saw above, we have

$$d(\xi_2 \mid \xi_2) = \xi_1^2 \mid \xi_1 \mid \xi_2 + \xi_2 \mid \xi_1^2 \mid \xi_1.$$

We would like to commute h_{20} past h_0 and then past h_1. In $C^\bullet_{E^0A(1)_*}(\mathbb{F}_2)$ we have the corresponding element

$$\xi_{1,1} \mid \xi_{1,0} \mid \xi_{2,0} + \xi_{2,0} \mid \xi_{1,1} \mid \xi_{1,0}.$$

From the previous example, we see that we have in $C^\bullet_{E^0A(1)_*}(\mathbb{F}_2)$

$$d(\xi_{1,1} \mid \xi_{1,0}\xi_{2,0}) = \xi_{1,1} \mid \xi_{1,0} \mid \xi_{2,0} + \xi_{1,1} \mid \xi_{2,0} \mid \xi_{1,0}$$

and

$$d(\xi_{1,1}\xi_{2,0} \mid \xi_{1,0}) = \xi_{1,1} \mid \xi_{2,0} \mid \xi_{1,0} + \xi_{2,0} \mid \xi_{1,1} \mid \xi_{1,0}.$$

Thus we see that

$$d(\xi_{1,1} \mid \xi_{1,0}\xi_{2,0} + \xi_{1,1}\xi_{2,0} \mid \xi_{1,0}) = \xi_{1,1} \mid \xi_{1,0} \mid \xi_{2,0} + \xi_{2,0} \mid \xi_{1,1} \mid \xi_{1,0}.$$

This is the reason $h_1h_0h_{2,0} = h_{20}h_1h_0$ in $C^\bullet_{E^0A(1)_*}(\mathbb{F}_2)$.

To get a May differential we lift this to $C^\bullet_{A(1)_*}(\mathbb{F}_2)$. Thus we want to add the elements $\xi_1^2 \mid \xi_1\xi_2$ and $\xi_1^2\xi_2 \mid \xi_1$ to $\xi_2 \mid \xi_2$. We need to calculate these the cobar differential on these elements. Note that

$$\psi(\xi_1\xi_2) = (\xi_1 \mid 1 + 1 \mid \xi_1)(\xi_2 \mid 1 + \xi_1^2 \mid \xi_1 + 1 \mid \xi_2).$$

Calculating this out shows that

$$d(\xi_1\xi_2) = \xi_1^3 \mid \xi_1 + \xi_1 \mid \xi_2 + \xi_2 \mid \xi_1 + \xi_1^2 \mid \xi_1^2.$$

An analogous calculation shows that

$$d(\xi_1^2\xi_2) = \xi_1^2 \mid \xi_2 + \xi_2 \mid \xi_1^2 + \xi_1^2 \mid \xi_1^3.$$

Remark 3.42. In $C^\bullet_{A_*}(\mathbb{F}_2)$ one would get

$$d(\xi_1^2\xi_2) = \xi_1^4 \mid \xi_1 + \xi_1^2 \mid \xi_2 + \xi_2 \mid \xi_1^2 + \xi_1^2 \mid \xi_1^3.$$

But in $A(1)_*$, $\xi_1^4 = 0$, so the first term does not appear.

Thus, we find that

$$d(\xi_1^2 \mid \xi_1\xi_2) = \xi_1^2 \mid \xi_1^3 \mid \xi_1 + \xi_1^2 \mid \xi_1 \mid \xi_2 + \xi_1^2 \mid \xi_2 \mid \xi_1 + \xi_1^2 \mid \xi_1^2 \mid \xi_1^2$$

and

$$d(\xi_1^2\xi_2 \mid \xi_1) = \xi_1^2 \mid \xi_2 \mid \xi_1 + \xi_2 \mid \xi_1^2 \mid \xi_1 + \xi_1^2 \mid \xi_1^3 \mid \xi_1.$$

Combining all of this together, we get that

$$d(\xi_2 \mid \xi_2 + \xi_1^2 \mid \xi_1\xi_2 + \xi_1^2\xi_2 \mid \xi_1) = \xi_1^2 \mid \xi_1^2 \mid \xi_1^2.$$

Now note that the May filtration of $\xi_2 \mid \xi_2$ is 6, where as the May filtration of $\xi_1^2\xi_2 \mid \xi_1$ and $\xi_1^2 \mid \xi_1\xi_2$ are both 4. Thus $\xi_2 \mid \xi_2$ and $\xi_2 \mid \xi_2 + \xi_1^2 \mid \xi_1\xi_2 + \xi_1^2\xi_2 \mid \xi_1$ give the same element in $C^\bullet_{E^0A(1)_*}(\mathbb{F}_2)$. Since the May filtration of $\xi_1^2 \mid \xi_1^2 \mid \xi_1^2$ is 3, this shows the following proposition.

Proposition 3.43. *In the May spectral sequence for $A(1)_*$, there is a d_3-differential*

$$d_3(b_{20}) = h_1^3.$$

Remark 3.44. Observe that in A_*, the element ξ_1^4 is non-zero. So we would obtain the following differential in the May SS for A_*, $d_3(b_{20}) = h_1^3 + h_0^2h_2$.

This shows the following.

Proposition 3.45. *The E_4-page of the May SS for $A(1)_*$ is*

$$P(h_0, h_1, \alpha, \beta)/(h_0h_1, \alpha^2 - h_0^2\beta, h_1\alpha)$$

where α is represented by h_0b_{20} and β is represented by b_{20}^2. Moreover, there are no higher May differentials and so this is also the E_∞-page.

Recall that $H^*(ko) = A/\!/A(1)$, and so the E_2-page of the Adams spectral sequence for ko is given by

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_2) \implies \pi_*(ko) \otimes \mathbb{Z}_2.$$

So the previous proposition gives us the E_2-term for the ASS for ko. Its easy to see from the chart that there are no possible Adams differentials. So the Adams spectral sequence collapses immediately.

I want to point out a couple of things. Note that there is the quotient map $A_* \to A(1)_*$ and hence a morphism of cochain complexes

$$C^\bullet_{A_*}(\mathbb{F}_2) \to C^\bullet_{A(1)_*}(\mathbb{F}_2)$$

Observe that this morphism is compatible with the May filtration on both sides, and hence we get a morphism of May spectral sequences. The morphism induces the obvious quotient map on E_1-pages

$$P(h_{i,j} \mid i \geq 1, j > 0) \to P(h_0, h_1, h_{20}).$$

Since the quotient map $A_* \to A(1)_*$ is an isomoprhism in degrees $t \leq 3$, we see that the map on E_1-pages above is an isomorphism in $t-s \leq 2$. For A_* there was the differential $d_3(b_{20}) = h_1^3 + h_0^2 h_2$, so since h_2 projects to 0 this shows the map is an epimorphism on the E_2-page in $t-s=3$. Thus the map

$$\mathrm{Ext}_{A_*}(\mathbb{F}_2) \to \mathrm{Ext}_{A(1)_*}(\mathbb{F}_2)$$

is an isomorphism for $t-s \leq 2$ and an epimorphism in $t-s=3$.

3.4. **Adams Vanishing and Periodicity.** In this section I want to sketch the argument of the Adams vanishing line of $\mathrm{Ext}_A(\mathbb{F}_p, \mathbb{F}_p)$ and also say something about the periodicity operators. I am also very lazy, so I won't consider all primes. Rather, I will focus on the case $p=2$. The material of this subsection can be found in [15], but was originally done by Adams in [3]. The odd primary version was originally done by Liulevicius in [9], but the techniques of [3] carry over.

Theorem 3.46. [3] *The groups* $\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ *are zero provided that* $0 < t-s < f(s)$ *where* $f(s) = 2s - \varepsilon$ *where*

$$\varepsilon = \begin{cases} 1 & s \equiv 0,1 \mod 4 \\ 2 & s \equiv 2 \mod 4 \\ 3 & s \equiv 3 \mod 4. \end{cases}$$

Remark 3.47. I have expressed the numerical function $f(s)$ following Ravenel. Adams originally wrote this theorem with $s > 0$ and $t < U(s)$ for a numerical function $U(s)$. We can go between the two statements via $f(s) = U(s) - s$.

The idea behind Adams' proof is to first consider the cofibre sequence

$$S^0 \to H\mathbb{Z} \to \overline{H}. \tag{3.48}$$

This gives a long exact sequence in (co)homology, but it actually turns out to be a short exact sequence. This gives a long sequence in Ext.

$$\cdots \to \mathrm{Ext}_A^{s,t}(\mathbb{F}_2) \to \mathrm{Ext}_A^{s,t}(A /\!\!/ A(0)_*) \to \mathrm{Ext}_A^{s,t}(H_*\overline{H}) \to \mathrm{Ext}_A^{s+1,t}(\mathbb{F}_2) \to \cdots$$

However, by a change-of-rings argument, we find that

$$\mathrm{Ext}_A(A /\!\!/ A(0)_*) \cong \mathrm{Ext}_{A(0)_*}(\mathbb{F}_2) \cong \mathbb{F}_2[v_0]$$

is all concentrated in $t-s=0$. Moreover, the map

$$\mathrm{Ext}_A^{s,t}(\mathbb{F}_2) \to \mathrm{Ext}_{A(0)_*}^{s,t}(\mathbb{F}_2)$$

is an isomorphism when $t-s=0$. This tells us that when $t-s>0$, then

$$\mathrm{Ext}_A^{s,t}(\mathbb{F}_2) \cong \mathrm{Ext}_A^{s-1,t}(H_*\overline{H}).$$

So it suffices to prove a vanishing line for $H_*\overline{H}$. Note that the cofibre sequence (3.48) gives a short exact sequence in homology

$$0 \to \mathbb{F}_2 \to A /\!\!/ A(0)_* \to \overline{A /\!\!/ A(0)}_* \to 0$$

We will let L stand for an A_*-comodule which is $A(0)_*$-free and connected. Define a numerical function

$$T(4k+i) := \begin{cases} 12k & i=0 \\ 12k+2 & i=1 \\ 12k+4 & i=2 \\ 12k+7 & i=3 \end{cases}$$

Theorem 3.49 (Adams' Vanishing theorem, [3])**.** $\mathrm{Ext}_{A(r)_*}^{s,t}(L)=0$ *for* $t<T(s)$*.*

We will prove this in a series of lemmas. In the following we shall want to consider $1 \le r \le \infty$, and we should understand $A(\infty)$ to be the Steenrod algebra A.

Lemma 3.50. *The Vanishing Theorem is true when* $r=\infty$ *and* $L=A(0)_*$ *and for* $s \le 4$*.*

Proof. This lemma is computational and has been left as an exercise. The key ingredient is to use the fact that $A(0)_* = H_*S/2$ and that in the derived category $\mathscr{D}_{A_*}$ there is a cofibre sequence

$$\mathbb{F}_2[-1] \xrightarrow{\cdot h_0} \mathbb{F}_2 \longrightarrow A(0)_*$$

Alternatively, one can write down a minimal resolution up through cohomological degree 4. $\square$

Lemma 3.51. *The Vanishing theorem is true when* $r=\infty$ *and* $s \le 4$ *for any* $A(0)$*-free* L*.*

Proof. As L is $A(0)$-free, it has a basis as an $A(0)$-module. This induces a filtration on L by the degree of the basis generators. That is, we set F_nL to be the sub-$A(0)$-module of L spanned by generators in degree $\geq n^2$. We will first prove that the statement holds for the quotients L/F_nL for all n. We do this by induction. We will then give a comparison to L itself.

The induction begins with L/F_0L, but since $F_0L = L$, we have that this quotient is 0, in which case the statement is vacuous. The filtration induces short exact sequences

$$0 \to F_nL/F_{n+1}L \to L/F_{n+1}L \to L/F_nL \to 0$$

and note that $F_nL/F_{n+1}L$ is a free $A(0)$-module with generators all in the same degree. This short exact sequence induces a long exact sequence in Ext,

$$\cdots \to \mathrm{Ext}_A^{s,t}(L/F_nL) \to \mathrm{Ext}_A^{s,t}(L/F_{n+1}L) \to \mathrm{Ext}_A^{s,t}(F_nL/F_{n+1}L) \to \cdots.$$

This is a bit confusing. Adams is working in modules, but I was originally in comodules and now we are in modules...I guess I dualized at some point...

By the previous lemma, the right hand Ext-group satisfies the vanishing theorem, and by induction it is true for the left hand Ext-group. Thus for $t < T(s)$, we have that the middle Ext-group is 0. Thus, we have shown that the statement holds for all of the quotients L/F_nL.

On the other hand, we have

$$\mathrm{Ext}_A^{s,t}(L) \cong \mathrm{Ext}_A^{s,t}(L/F_nL)$$

for $n \gg 0$. This proves the lemma. □

At this point, we need to introduce *Margolis homology*.

Suppose that L is a module over A. Then, in particular, L is a module over $A(0) = E(\mathrm{Sq}^1)$. Since Sq^1 squares to 0, we can regard it as defining a differential on L.

Definition 3.52. The Sq^1-Margolis homology of L is

$$M_*(L;\mathrm{Sq}^1) := \frac{\ker \mathrm{Sq}^1}{\mathrm{im}\,\mathrm{Sq}^1}.$$

The following is fairly straightforward to see.

Proposition 3.53. *An $A(0)$-module L is free if and only if $M_*(L;\mathrm{Sq}^1) = 0$.*

Lemma 3.54. *Let*

$$0 \to L' \to L \to L'' \to 0$$

be a short exact sequence of $A(0)$-modules. Then if two of three of them are $A(0)$-free, then so is the third.

[2]We take this convention since, when we think of L as an $A(0)$-module, acting by Sq^1 increases the degree.

Proof. This follows immediately from the LES in Margolis homology. □

Proposition 3.55. *The Vanishing Theorem holds in the case $r = \infty$.*

Proof. Let L be an A-module which is free over $A(0)$ and connected. Form the first four terms in a minimal A-free resolution of L, so

$$P_4 \xrightarrow{d_4} P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} L.$$

As A is itself $A(0)$-free, it follows from the previous lemma that the images of the d_i are $A(0)$-free for all i. Let M denote the image of d_4. Then by Lemma 3.51, we have that $M_t = 0$ for $t < 12$.

We will now induct on k. That is, we will show by induction on k that for any $j \leq k$, and $t < T(4j+i)$, $i = 0,1,2,3$, one has $\mathrm{Ext}_A^{s,t}(L) = 0$ for any $A(0)$-free A-module L. By Lemma 3.51, the case $k = 0$ holds for any such L. Suppose that the statement holds for some k. We may then apply the inductive hypothesis to M. Since we have an isomorphism

$$\mathrm{Ext}_A^{s+4,t}(L) \cong \mathrm{Ext}_A^{s,t}(M)$$

we can conclude that the result is true for L whenever $t < T(4j+i)$, but now we may allow for $j = k+1$. This completes the induction. □

In order to deduce the Vanishing Theorem, we need to allow r to be any natural number. Observe that it is trivial in the case $r = 0$. So assume that $0 < r < \infty$. The idea is to use the change of rings isomorphism

$$\mathrm{Ext}_A^{s,t}(A \otimes_{A(r)} L) \cong \mathrm{Ext}_{A(r)}^{s,t}(L)$$

This allows us to try and use the already known case of the Vanishing Theorem (when $r = \infty$) to the module $A \otimes_{A(r)} L$. In order to do this, we need to know that $A \otimes_{A(r)} L$ is still free as an $A(0)$-module. It turns out to be easier to show that $A(\rho) \otimes_{A(r)} L$ is free as an $A(0)$ module for any $\rho \geq r$. Since L is free as an $A(0)$-module, it is sufficient to prove that $A(\rho) \otimes_{A(r)} A(0)$ is free as an $A(0)$-module.

Lemma 3.56. *The module $A(\rho) \otimes_{A(r)} A(0)$ is free as an $A(0)$-module.*

Not sure if this actually works, but it was an interesting idea, so I wrote it up to double check later.

Proof.
Regard everything in sight as a cochain complex via multiplication by Sq^1. Then we can regard the tensor product above as a tensor product in cochain complexes (Is that part right?). We then get a Tor-spectral sequence

$$Tor^{M_*(A(r);\mathrm{Sq}^1)}(M_*(A(\rho);\mathrm{Sq}^1), M_*(A(0);\mathrm{Sq}^1)) \Longrightarrow M_*(A(\rho) \otimes_{A(r)} A(0);\mathrm{Sq}^1).$$

The input is clearly 0, and so the answer is 0. This shows that the module is free as an $A(0)$-module. □

From this the Vanishing Theorem follows. From the Vanishing Theorem, we can prove a (slightly) weaker version of Adams' vanishing line.

Definition 3.57. Let $V(s)$ be the numerical function defined by

$$V(4k+i)=\begin{cases}12k-3 & i=0\\ 12k+2 & i=1\\ 12k+4 & i=2\\ 12k+6 & i=3\end{cases}$$

Theorem 3.58 (Vanishing Theorem Lite). *We have that* $\mathrm{Ext}_A^{s,t}(\mathbb{F}_2)=0$ *provided that* $0<s<t<V(s)$.

Proof. As we saw above, we have an isomorphism

$$\mathrm{Ext}_A^{s,t}(\mathbb{F}_2)\cong \mathrm{Ext}_A^{s-1,t}(\overline{A/\!\!/A(0)}_*)$$

Since $\overline{A/\!\!/A(0)}$ is $A(0)$-free we can apply the Vanishing Theorem to this module. Thus we can conclude that

$$\mathrm{Ext}_A^{s-1,t}(\overline{A/\!\!/A(0)}_*)=0$$

so long as

$$0<s-1<t<T(s-1)+2.$$

The extra 2 arises because the connectivity of $\overline{A/\!\!/A(0)}$ is 2, and so the Vanishing Theorem actually applies to $\Sigma^{-2}A/\!\!/A(0)$. Observe that $T(s-1)+2=V(s)$. □

Remark 3.59. This is only a slightly weaker version of the the desired vanishing line result. The way to see this is to let $g(s):=V(s)-s$. Then we have that $g(s)=2s-\eta(s)$ where

$$\eta(s)=\begin{cases}3 & s\equiv 0,3 \mod 4\\ 1 & s\equiv 1 \mod 4\\ 2 & s\equiv 2 \mod 4\end{cases}$$

In order to get a more optimal result, Adams has to do quite a bit of extra work to obtain his periodicity theorem.

3.5. **Other computational techniques.** I want to briefly sketch other techniques one can use to do calculations in the May spectral sequence. These rely on additional structure on Ext_A. This is in fact, quite general, and we give the following theorem of May (cite May's "general algebraic approach...").

Theorem 3.60. *Let Γ be a Hopf algebra over $\mathbb{Z}/2$ and let N be a left Γ-comodule algebra. Then* $\mathrm{Ext}_\Gamma(\mathbb{F}_2, N)$ *carries* algebraic Steenrod operations

$$\mathrm{Sq}^n : \mathrm{Ext}_\Gamma^{s,t}(N) \to \mathrm{Ext}_\Gamma^{s+n,2t}(\mathbb{F}_p)$$

which satisfies the following for $x \in \mathrm{Ext}^{s,t}$,

1. $\mathrm{Sq}^i(x) = 0$ *if* $i > s$,
2. $\mathrm{Sq}^i(x) = x^2$,
3. *Cartan formula*
4. *Adem relations*

May showed the following in his thesis.

Proposition 3.61. *Let* $x \in \mathrm{Ext}_\Gamma^{s,t}(N)$ *be represented in the cobar complex by a sum of elements of the form* $\gamma_1 \mid \ldots \mid \gamma_s n$. *Then* $\mathrm{Sq}^0(x)$ *is represented by a similar sum of the form* $\gamma_1^2 \mid \ldots \mid \gamma_s^2 n^2$.

The up shot of this is that in the case of the dual Steenrod algebra, we have

$$\mathrm{Sq}^0(h_{i,j}) = h_{i,j+1}.$$

Nakaura showed that there is an intricate relationship between the Squaring operations and the cobar differential. First, keep in mind that in order to get the algebraic Steenrod operations, one first defines operations

$$\mathrm{Sq}_i : C^\bullet(A_*) \to C^\bullet(A_*)$$

by

$$\mathrm{Sq}_i(x) = x \cup_i x + \delta(x) \cup_{i+1} x$$

in the usual way.

Theorem 3.62 (Nakamura). $\mathrm{Sq}_{i+1}\delta = \delta \mathrm{Sq}_i$ *for* $i \geq 0$.

This translates into differentials in the May spectral sequence. In general, what one has is formulas of the following type,

$$d_?(\mathrm{Sq}^n x) = \mathrm{Sq}^n d_r x.$$

So if you can find a differential on some element x, then one can obtain further differentials in the May spectral sequence using squaring operations. Let's see this in an example.

Example 3.63. We have the May d_1-differential $d_1(h_{20}) = h_1 h_0$. Then, via Nakamura's theorem, we have

$$d_?(b_{20}) = d_?(\mathrm{Sq}^1 h_{20}) = \mathrm{Sq}^1 d_1(h_{20}) = \mathrm{Sq}^1(h_1 h_0),$$

by the Cartan formula we find that

$$\mathrm{Sq}^1(h_1 h_0) = \mathrm{Sq}^1(h_1)\mathrm{Sq}^0 h_0 + \mathrm{Sq}^0(h_1)\mathrm{Sq}^1(h_0) = h_1^2 h_1 + h_2 h_0^2 = h_1^3 + h_0^2 h_2,$$

which is what we found explicitly before. Note that by comparison of May weights we can figure out that ? = 3. Applying Sq^0 over and over, we also find

$$d_3(b_{2,j}) = h_{j+1}^3 + h_j^2 h_{j+2}.$$

Let's do another example.

Example 3.64. We have that $d_3(b_{20}^2) = 0$. Thus we should try to use Nakamura's theorem. We have

$$d_?(b_{20}^2) = d_?(\mathrm{Sq}^2(b_{20})) = \mathrm{Sq}^2 d_3(b_{20}) = \mathrm{Sq}^2(h_1^3 + h_0^2 h_2).$$

Using the Cartan formula, one can check that

$$\mathrm{Sq}^2(h_1^3) = 0$$

and

$$\mathrm{Sq}^2(h_0^2 h_2) = h_0^4 h_3.$$

Thus we have $d_7(b_{20}^2) = h_0^4 h_3$.

Again, iteratively using Sq^0, we find that $d_7(b_{2,j}^2) = h_j^4 h_{j+3}$.

In general, one will find that

$$d_{2^{i+1}-1}(b_{20}^{2^i}) = h_0^{2^{i+1}} h_{i+2}.$$

for $i \geq 1$.

Example 3.65.

3.6. **An Adams differential.** I now want to sketch an argument given by Adams in [1], and use it to derive an Adams differential on h_4. First, I need to recall some important relations in the Adams E_2-term for the sphere.

Theorem 3.66. *For $p = 2$, we have*

1. *(Adams [2])* Ext^2 *is spanned by the $h_i h_j$ where $0 \leq i \leq j$ and $j \neq i+1$,*
2. *(Wang)* Ext^3 *is spanned by $h_i h_j h_k$ subject to the relations*
   - $h_i h_{i+1} h_k = 0$,
   - $h_i^2 h_{i+2} = h_{i+1}^3$,
   - $h_i (h_{i+2})^2 = 0$

   along with the elements

$$c_i = \langle h_{i+1}, h_i, h_{i+2}^2 \rangle \in \mathrm{Ext}^{3,11\cdot 2^i}.$$

We will deduce from this the following.

Theorem 3.67 (Adams [1]). *If $\pi_{2n-1}S^n$ and $\pi_{4n-1}S^{2n}$ both contain elements of Hopf invariant one, then $n \leq 4$.*

Recall that the h_i are exactly the elements in the Adams E_2-term which could detect elements of Hopf invariant 1. Thus, what needs to be shown is that it is not possible for h_m and h_{m+1} to both be permanent cycles of the Adams spectral sequence unless $m < 4$.

To prove the theorem, suppose towards a contradiction that h_m and h_{m+1} and $m \geq 3$. Consider the element $h_0h_m^2 \in \mathrm{Ext}_A^{3,2^{m+1}+1}(\mathbb{F}_2)$. Then by the above theorem, this element is non-zero, and it is a d_2-cycle as it is a product of d_2-cycles. It is also not a boundary under d_2. Indeed, since the 1-line is spanned by the h_i, the d_2-boundaries in the 2-line are spanned by $d_2(h_i)$. Note that since $h_i \in \mathrm{Ext}^{1,2^i}$, the boundary $d_2(h_i)$ is in $\mathrm{Ext}^{3,2^i+1}$. In particular, it could be the case that $d_2(h_{m+1}) = h_0h_m^2$, since the bidegrees work out correctly. However, this doesn't happen since we assumed that h_m and h_{m+1} are permanent cycles. Thus h_{m+1} doesn't support a d_2-differential. But if $h_0h_m^2$ were going to die in the spectral sequence, then it would have to be killed by a d_2-differential, and as this differential doesn't occur, it follows that it is not killed in the spectral sequence. As h_m is also a permanent cycle, $h_0h_m^2$ doesn't support a differential. Thus $h_0h_m^2$ is a non-zero element in $E_\infty^{3,2^{m+1}+1}$. Let h_i' denote the class in $\pi_{2^i-1}S^0$ which is detected by h_i. Then as h_m' lives in odd stem, we have $2(h_m')^2 = 0$. But this implies that $h_0h_m^2 = 0$ in E_∞. This is a contradiction. This proves the theorem.

The argument above gives us the following corollary.

Corollary 3.68. *We have the Adams d_2-differential $d_2(h_4) = h_0h_3^2$.*

4. Complex-oriented cohomology theories, Quillen's Theorem, and Brown-Peterson theory

4.1. Complex-oriented Cohomology Theories.

Throughout let E be a ring spectrum.

Definition 4.1. A *complex orientation* of E is an element $x \in \widetilde{E}^*(\mathbb{C}P^\infty)$ such that under the restriction map

$$\widetilde{E}^*(\mathbb{C}P^\infty) \xrightarrow{i^*} \widetilde{E}^*(\mathbb{C}P^1) = \widetilde{E}^*(S^2)$$

the class x is mapped to a generator. Here, we regard $\widetilde{E}^*(\mathbb{C}P^1)$ as a free rank 1 module over π_*E.

Remark 4.2. In light of the suspension isomorphism

$$\widetilde{E}^*(S^0) \cong \widetilde{E}^{*+2}(S^2)$$

we have a canonical generator x_{can} of $\widetilde{E}^*(S^2)$ corresponding to 1. Thus, a complex orientation x has the property that $i^*x = ux_{can}$ for some unit $u \in E_*^\times$. In general, one has to keep track of this u in formulas. This is one of the reasons people use the more rigid definition that $i_*x = x_{can}$, i.e. $u = 1$. This way, one does not have to keep track of u's in various formulas.

Example 4.3. The Eilenberg-MacLane spectrum HR is complex oriented for any ring R, since HR.

Example 4.4. If $E = KU$, then $\pi_*KU \cong \mathbb{Z}[\beta^\pm]$ where $|\beta| = 2$. Recall that the isomorphism $\pi_2 KU \to \widetilde{K}^0(S^2)$ sends β to the class $1-L$ where L is the Hopf bundle over $\mathbb{C}P^1$, i.e. the tautological bundle over $\mathbb{C}P^1$. Thus a complex orientation for KU is the class $x = \beta^{-1}(1-\gamma) \in \widetilde{K}^2(\mathbb{C}P^\infty)$, where γ is the tautological bundle.

Example 4.5. Let MU be the complex cobordism spectrum. Then

$$MU(1) = (\mathbb{C}P^\infty)^\gamma.$$

I claim that this Thom space is equivalent to $\mathbb{C}P^\infty$. Indeed, let $E(\gamma)$ denote the total space of γ. This bundle can be given a metric, so we can consider the corresponding disc bundle $D(\gamma)$ and S^1-bundle $S(\gamma)$. Note that $S(\gamma)$ is a principal $U(1)$-bundle, in fact it is the universal $U(1)$-bundle. Thus $S(\gamma)$ is contractible. The Thom space, in this case, is then

$$(\mathbb{C}P^\infty)^\gamma = D(\gamma)/S(\gamma).$$

As $S(\gamma)$ is contractible, the quotient map

$$E(\gamma) \to (\mathbb{C}P^\infty)^\gamma$$

is an equivalence, and hence the zero section

$$\zeta : \mathbb{C}P^\infty \to E(\gamma) \to (\mathbb{C}P^\infty)^\gamma$$

is an equivalence. A complex orientation for MU is then class given by the map

$$\Sigma^{-2}\mathbb{C}P^\infty \xrightarrow{\zeta} \Sigma^{-2}MU(1) \to MU.$$

To see that this restricts to a generator, note that the map

$$\Sigma^2 MU(0) \to MU(1)$$

used in defining MU is the inclusion of $\mathbb{C}P^1$ into $\mathbb{C}P^\infty$.

Recall that the Atiyah-Hirzebruch spectral sequence is a method to calculate E^*X from the singular cohomology of X,

$$E_2^{p,q} = H^p(X; \pi_q E) \implies E^{p+q}X.$$

This is a multiplicative spectral sequence and the differentials are linear over E_*. There are also reduced versions and homological versions. See Kochman for a thorough construction.

We use the AHSS to calculate $E^*(\mathbb{C}P^\infty)$ and related groups.

Proposition 4.6. *We have the following:*

(1) $E^*\mathbb{C}P^n \cong E^*[x]/(x^{n+1})$ *where x is the restriction of the complex orientation to $\mathbb{C}P^n$,*
(2) $E^*\mathbb{C}P^\infty \cong E^*[[x]]$ *where x is the complex orientation,*
(3) $E^*(\mathbb{C}P^n \times \mathbb{C}P^m) \cong E^*[x_1, x_2]/(x_1^{n+1}, x_2^{m+1})$, *and*
(4) $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[x_1, x_2]]$.

Proof. We utilize the Atiyah-Hirzebruch spectral sequence. Consider

$$E_2 = H^*(\mathbb{C}P^n; E_*) \implies E^*(\mathbb{C}P^n).$$

Now since $H^*(\mathbb{C}P^n)$ is a free abelian group, we have that the

$$E_2 \cong H^*(\mathbb{C}P^n) \otimes E_* \cong E_*[c_1]/(c_1^{n+1}).$$

Now the class $x \in E^*\mathbb{C}P^n$ corresponding to the complex orientation is detected by an element t in the E_2-term. Note further that the AHSS is natural in X. Thus, the map $\mathbb{C}P^1 \to \mathbb{C}P^n$ induces a morphism of Atiyah-Hirzebruch spectral sequences. Clearly, the AHSS for $\mathbb{C}P^1$ collapses at E_2. The assumptions on the complex orientation x implies that i^*x is a generator of $\pi_2 E$. This is reflected in the E_2-page of the AHSS by saying that on the AHSS E_2-term, $i * t$ is a generator of the E_2-term for $\widetilde{E}^*(\mathbb{C}P^1)$. This implies that $i * t$ is the class c_1 up to a unit in $\pi_* E$. This implies that the class c_1 does not support a differential. So as the spectral sequence is multiplicative and linear over E_*, it follows that it collapses at E_2.

Now it could be the case that x^{n+1} is not zero, but rather some polynomial of degree at most n. To exclude this possibility, note that $x \in \widetilde{E}^*(\mathbb{C}P^n) = E(\mathbb{C}P^n, *)$. Let U_i be the affine open consisting of points $[x_1, \ldots, x_{i-1}, 1, \ldots, x_{n+1}]$. As U_i is contractible, it follows that $x \in E^*(\mathbb{C}P^n, U_i)$ for each i. Thus

$$x^{n+1} \in E^*(\mathbb{C}P^n, U_1 \cup \cdots \cup U_{n+1}) = E^*(\mathbb{C}P^n, \mathbb{C}P^n) = 0.$$

Hence $x^{n+1} = 0$. This shows that $E^*(\mathbb{C}P^n) = E^*[x]/(x^{n+1})$.

We obtain the second statement by using the Milnor exact sequence

$$0 \to \lim^1 E^*(\mathbb{C}P^n) \to E^*(\mathbb{C}P^\infty) \to \lim E^*(\mathbb{C}P^n) \to 0.$$

It is clear via the Atiyah-Hirzebruch spectral sequence that the maps

$$E^*(\mathbb{C}P^{n+1}) \to E^*(\mathbb{C}P^n)$$

is the obvious projection. So the Mittag-Leffler condition holds and hence $\lim^1$ vanishes. The last two statements are proved analogously. □

One can also use the Atiyah-Hirzebruch spectral sequence to prove the following.

Proposition 4.7. *If ξ is a complex bundle over X, then $\widetilde{E}^*(\mathbb{P}(\xi))$ is a free $E^*(X)$ module on the generators $1, x, \ldots, x^n$ coming from the cohomology of the fibre $\mathbb{C}P^n$.*

Proof. This is from the generalized version of the Leray-Hirsch theorem. See Switzer 15.47 for details. □

Once you have this, you can define chern classes c_i for the cohomology theory E. Indeed, the chern classes for ξ can be defined to be the elements $c_i(\xi) \in E^{2i}(X)$ such that

$$x^{n+1} = (-1)^{n+1} c_n(\xi) \cdot 1 + (-1)^n c_{n-1}(\xi) x + \cdots + c_1(\xi) \cdot x^{n-1}.$$

Note that for the above to work we need to have $x \in E^2(\mathbb{C}P^\infty)$. For more details see Switzer 16.2. In particular, we can think of x as the first Chern class of the tautological bundle.

Now the space $\mathbb{C}P^\infty$ classifies complex line bundles. That is a complex line bundle on a space X (up to isomorphism) is the same as a homotopy class of maps

$$X \to \mathbb{C}P^\infty.$$

Now if L_1 and L_2 are two line bundles, we can tensor them together to obtain a new line bundle $L_1 \otimes L_2$. This in fact defines a group structure on the set $Pic(X)$ of complex line bundles on X. Since $Pic(X)$ is represented by $\mathbb{C}P^\infty$, this structure must be given by maps

$$m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

This, in turn, induces a map

$$E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty).$$

By the above, this is just a map

$$E^*[[x]] \to E^*[[x_1, x_2]].$$

Thus, x is mapped to a power series $F(x_1, x_2)$. Now the tensor product $\otimes$ on line bundles satisfies several properties: unitality, associativity, commutativity. These translate into properties of $F(x_1, x_2)$. In particular, they imply

(1) $F(0, x) = F(x, 0) = x$,
(2) $F(F(x, y), z) = F(x, F(y, z))$,
(3) $F(x, y) = F(y, x)$.

Now if L is a line bundle on X, its dual bundle $L^\vee$ is an inverse of L under the tensor product

$$L \otimes L^\vee \cong \varepsilon.$$

The dualization of a line bundle is represented by a map

$$\mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

and hence we have a map

$$\iota : E^*\mathbb{C}P^\infty \to E^*\mathbb{C}P^\infty.$$

This has the property

$$F(x, \iota x) = 0.$$

This is an example of a formal group law.

Definition 4.8. A *1-dimensional commutative formal group law* over a (graded) ring R is a (homogenous) element of $R[[x_1, x_2]]$ which satisfies the properties above. That is, it is a pair (F, ι) where $F \in R[[x_1, x_2]]$ and $\iota \in R[[x]]$ such that the following properties hold

(1) $F(0, x) = F(x, 0) = x$,
(2) $F(F(x, y), z) = F(x, F(y, z))$,
(3) $F(x, y) = F(y, x)$, and
(4) $F(x, \iota x) = 0$.

We call ι the *formal inverse*.

Observe that these properties imply that

$$F(x_1, x_2) = x_1 + x_2 + \sum_{i,j>0} a_{ij} x_1^i x_2^j.$$

Indeed, we have a priori that

$$F(x_1, x_2) = \sum_{i,j \geq 0} a_{ij} x_1^i x_2^j$$

for $a_{ij} \in R$. Then property (1) shows that $a_{00} = 0$,

$$a_{i0} = \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 \end{cases}$$

and

$$a_{0j} = \begin{cases} 1 & j = 1 \\ 0 & j \neq 1 \end{cases}$$

Observe that property (3) implies the following symmetry

$$a_{ij} = a_{ji}$$

for all i and j. The second relation (2) implies a great number of identities amongst the a_{ij}. ...

write down example...

Example 4.9. Let $E = H\mathbb{Z}$. Then $m^*(x) = x_1 + x_2$.

Example 4.10. If $E = KU$ then we have $x = \beta^{-1}(1-\gamma)$. Recall that the map

$$m : \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$$

was defined so that it represents the tensor product of line bundles, that is

$$m^*(\gamma) = \pi_1^*\gamma \otimes \pi_2^*\gamma.$$

This implies that in $K^*(\mathbb{C}P^\infty)$,

$$m^* x = \beta^{-1}(1 - \gamma \otimes \gamma).$$

Now observe that $\gamma = 1 - \beta x$. Let $x = \beta^{-1}(1-\gamma\otimes 1)$ and $y = \beta^{-1}(1-1\otimes\gamma)$ in $K^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. Then

$$\begin{aligned} \gamma \otimes \gamma &= (1-\beta x)(1-\beta y) \\ &= 1 - \beta y - \beta x + \beta^2 xy. \end{aligned}$$

Hence

$$m^*(x) = x + y - \beta xy.$$

This is what is called the *multiplicative formal group* $\widehat{\mathbb{G}}_m$.

Example 4.11. If $E = MU$, then we get a formal group law $F(x_1, x_2)$ where $a_{ij} \in \pi_{2(i+j-1)}(MU)$. This formal group law turns out to be rather complicated.

Remark 4.12. People, including me, often write $x +_F y$ instead of $F(x, y)$ for a formal group law F. When F arises from a complex oriented theory E.

We now remark on relating complex oriented theories. Suppose that E is a complex oriented theory with complex orientations x^E. Suppose further that $f : E \to F$ is a morphism of ring spectra. Then we can endow F with a complex orientation. Indeed, if $i : \mathbb{C}P^1 \to \mathbb{C}P^\infty$ is the standard inclusion, then $i^* x^E = u g^E$ where g^E is the canonical generator of $\widetilde{E}^2(\mathbb{C}P^1)$ and $u \in \pi_* E$ is a unit. Pushing forward via f gives a class $f_* x^E$ in $F^*(\mathbb{C}P^\infty)$ such that

$$i^* f_* x^E = f_* i^* x^E = f_*(u g^E) = f_*(u) g^F.$$

As u was a unit in $\pi_* E$, we have that $f_* u$ is a unit in $\pi_* F$. This shows that $f_* x^E$ is a complex orientation of F. Moreover, we find that

$$m^*(f_* x^E) = f_* m^*(x^E) = f_* F^E(x_1^E, x_2^E) = (f_* F^E)(f_* x_1^E, f_* x_2^E).$$

Here, we have written $f_* F^E$ for the formal power series

$$x + y + \sum_{i,j \geq 0} f_*(a_{ij}) x^i y^j.$$

In particular, this shows that $f_* x^E$ is a generator of $F^*(\mathbb{C}P^\infty)$.

Now, more often than not, the spectrum F will already be endowed with a complex orientation x^F. Now x^F is a also a generator of $F^*(\mathbb{C}P^\infty)$, and this generator will give rise to a different formal group law F^F (pardon the hideous notation). So how are $f_* F^E$ and F^F related? Well, since $f_* x^E$ and x^F are both generators for $F^*(\mathbb{C}P^\infty)$, we can express $f_* X^E$ as a formal power series in x^F. Thus, there are $c_i \in \pi_* F$ such that

$$f_* x^E = \sum_{i \geq 1} c_i (x^F)^i = \varphi(x^F)$$

for $\varphi(x) \in F_*[[x^F]]$. Since $f_* x^E$ also generates $F^* \mathbb{C}P^\infty$, it must be the case that $\varphi(x) \in F_*[[x^F]]^\times$ is an element of the group of units. This shows that

$$\frac{d}{dx^F} \varphi(x^F) \bigg|_{x^F = 0} = c_1$$

is a unit of $\pi_* F$. Note also that $f_* u^E = c_1 u^F$.

We have thus shown the following.

Lemma 4.13. *We have the identity*

$$g(F^F(x_1^F, x_2^F)) = (f_* F^E)(g(x_1^F), g(x_2^F)).$$

This leads us to a notion from the theory of formal group laws.

Definition 4.14. Let F and G be two formal group laws over a ring R. Then a *homomorphism* $\varphi : F \to G$ is a power series $\varphi \in R[[x]]$ such that

$$\varphi(F(x_1, x_2)) = G(\varphi(x_1), \varphi(x_2)).$$

We say that φ is an *isomorphism* if φ is invertible as an element of $R[[x]]$. This is the same as saying

$$\frac{d}{dx} \varphi(x) \bigg|_{x=0}$$

is a unit u. We say that φ is a *strict isomorphism* if $u = 1$.

Remark 4.15. Using the notation $x +_F y$, we can see why the above is called a homomorphism of formal group laws. It is the same as writing

$$\varphi(x +_F y) = \varphi(x) +_G \varphi(y).$$

Thus we can rephrase the previous lemma as follows.

Lemma 4.16. *Suppose that E and F are complex oriented ring spectra and $f : E \to F$ is a morphism of ring spectra. Then there is an isomorphism $g : f_* F^E \to F^F$ between the formal group laws determined by $f_* x^E$ and x^F.*

Corollary 4.17. *If E is a complex oriented ring spectrum then any two complex orientations of E give rise to isomorphic formal group laws.*

Remark 4.18. The above corollary can be thought of in the following terms. A formal group law requires a coordinate. But underlying a formal group law is an object called a *formal group*, it is analogous to the difference between a local chart of a Lie group around the identity element and the Lie group itself. So the above is saying that different complex orientations of E give rise to different formal group laws, but they are all presentations of the same underlying *formal group*. So to a *complex orientable* E we can associate a unique formal group $\widehat{\mathbb{G}}_E$.

We will now calculate the E-homology of $\mathbb{C}P^\infty$ and related spaces. First, I need to say something about the relationship between the homological and cohomological AHSS.

Proposition 4.19. *[cf. Kochman]Let E be a ring spectrum, let X be a CW complex. Consider the AHSS* add references

$$E^2_{p,q} = H_p(X, E_q) \implies E_{p+q}(X)$$

and

$$E_2^{p,q} = H^p(X; E_q) \implies E^{p+q}(X).$$

Then there is a natural pairing

$$\langle -,- \rangle : E_r^{n,-s} \otimes E^r_{n,t} \to E_{s+t}$$

such that

- *The pairing on $E_2 \otimes E^2$ is the usual one on singular (co)homology,*
- $\langle d_r x, y \rangle = \langle x, d^r y \rangle$,
- *the pairing on $E^*(X) \otimes E_* X$ induces the pairing on $E_\infty \otimes E^\infty$.*

We use this to show the following.

Proposition 4.20. *We have the following.*

(1) *The homological Atiyah-Hirzebruch spectral sequences for the E-homology of $\mathbb{C}P^n, \mathbb{C}P^\infty, \mathbb{C}P^n \times \mathbb{C}P^m$ and $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ all collapse at E_2.*
(2) *$E^*(\mathbb{C}P^n)$ and $E_*\mathbb{C}P^n$ are dual finitely generated free modules over E_*,*
(3) *There is a unique element $\beta_n \in E_*\mathbb{C}P^n$ such that*

$$\langle x^i, \beta_n \rangle = \delta_{in}$$

We write the image of the β_n in $E_*\mathbb{C}P^\infty$ by β_n as well.
(4) *$E_*\mathbb{C}P^n$ is free over E_* on $\beta_0, \beta_1, \ldots, \beta_n$. $E_*\mathbb{C}P^\infty$ is free on β_i for all $i \in \mathbb{N}$, and $E_*(\mathbb{C}P^n \times \mathbb{C}P^m)$ has as a basis $\beta_i\beta_j$ for $i \leq n$ and $j \leq m$. Same thing for $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$.*
(5) *The external product*

$$E_*\mathbb{C}P^\infty \otimes_{E_*} E_*\mathbb{C}P^\infty \to E_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

is an isomorphism.

Proof. The proof of (1) is straightforward. We have that $H_*(\mathbb{C}P^n) = \mathbb{Z}\{b_0, \dots, b_n\}$ where b_i is dual to x^i. Using the pairing, we have that each b_i is a permanent cycle since x^i is a permanent cycle. This proves (1) for $\mathbb{C}P^n$. We obtain the corresponding statement for $\mathbb{C}P^\infty$ via naturality, and the argument for the others is analogous. We obtain (2) and (3) immediately from the pairing between the AHSS. From part (1), we see that the E_2-term of the AHSS of each has a basis given by the β_i or $\beta_i\beta_j$. This gives rise to (4) and (5). □

Remark 4.21. Note that the β_i are dependent on the complex orientation x. When we want to emphasize this dependence, we shall write β_i^E.

Remark 4.22. Note that the pairing

$$E^*\mathbb{C}P^\infty \otimes_{E_*} E_*\mathbb{C}P^\infty \to E_*$$

induces a homomorphism

$$E^*(\mathbb{C}P^\infty) \to \hom_{E_*}(E_*\mathbb{C}P^\infty, E_*)$$

which, in this case, is an isomorphism. The correspondence works as follows. If $\sum a_n x_E^n \in E^*\mathbb{C}P^\infty$, then the corresponding map $E_*\mathbb{C}P^\infty \to E_*$ is determined by $b_n \mapsto a_n$.

Now, in light (5) of the last proposition, the diagonal map

$$\Delta : \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$$

endows $E_*\mathbb{C}P^\infty$ with a coalgebra structure. Let's recall how this works in singular (co)homology. We have the pairing

$$\langle \cdot, \cdot \rangle : H^*\mathbb{C}P^\infty \otimes H_*\mathbb{C}P^\infty \to \mathbb{Z}$$

and we have the adjunction

$$\langle \Delta(a \otimes b), \beta \rangle = \langle a \otimes b, \Delta_*\beta \rangle.$$

Note that Δ^* is how we get the cup product. So we have

$$\langle x^i \otimes x^j, \Delta_*\beta_k \rangle = \langle \Delta^*(x^i \otimes x^j), \beta_k \rangle = \langle x^{i+j}, \beta_k \rangle.$$

This shows that

$$\Delta_*\beta_n = \sum_{i+j=n} \beta_i \otimes \beta_j.$$

The same argument generalizes to complex oriented theories. Thus we have shown,

Proposition 4.23. *The coproduct on $E_*\mathbb{C}P^\infty$ is given by*

$$\Delta(\beta_n) = \sum_{i+j=n} \beta_i \otimes \beta_j.$$

Since β_i above is actually dependent on the complex orientation, we should see how they are transformed by morphisms of complex oriented ring spectra. Now suppose that we are in the situation as above: $f : E \to F$ is a morphism of ring spectra and E and F have complex orientations x^E and x^F respectively. As we saw before, there is a power series $\varphi(x)$ such that

$$\varphi(x^F) = f_* x^E,$$

so that

$$\varphi^{-1}(f_* x^E) = x^F$$

Let

$$\varphi^{-1}(x) = d_1 x + d_2 x^2 + d_3 x^3 + \cdots$$

where $d_i \in \pi_* F$. We also get that

$$(x^F)^j = \sum_i d_{i,j} (f_* x^E)^i$$

for some $d_{i,j} \in \pi_* F$. Then the usual pairing shows that

Lemma 4.24. $f_* \beta_i^E = \sum_j d_{i,j} \beta_j^F$.

Its also a good idea to try and determine $E_* BU$. The reason this is a good idea is because, since E is complex oriented, we have a Thom isomorphism for complex virtual bundles. Thus we get an isomorphism

$$E_* BU \cong E_* \mathrm{MU}.$$

Of particular interest to us when we look at the Adams-Novikov spectral sequence will be the case $E = \mathrm{MU}$.

Towards the end of calculating $E_* BU$, its a good idea to calculate $E_* BU(n)$. Now observe that there are maps

$$BU(n) \times BU(m) \to BU(n+m)$$

which represents the functor

$$\xi, \eta \mapsto \xi \oplus \eta$$

where ξ and η are rank n and m bundles respectively. These fit into a commutative diagram

$$\begin{array}{ccc} BU(n) \times BU(m) & \longrightarrow & BU(n+m) \\ \downarrow & & \downarrow \\ BU \times BU & \longrightarrow & BU \end{array}$$

where the bottom map represents the functor of direct sums of virtual vector bundles. These maps give rise to pairings

$$\mathrm{MU}(n) \wedge \mathrm{MU}(m) \to \mathrm{MU}(n+m)$$

and hence taking the colimits we get a map

$$\mathrm{MU} \wedge \mathrm{MU} \to \mathrm{MU}.$$

This makes MU into a ring spectrum. This is the structure which gives rise to ring structures on E_*BU and $E_*\mathrm{MU}$. Finally, there is a map

$$\vartheta_n : (\mathbb{C}P^\infty)^{\times n} \to BU(n)$$

which represents the functor

$$L_1, \dots, L_n \mapsto L_1 \oplus \cdots \oplus L_n$$

where L_i are line bundles. This induces a map

$$E_*(\mathbb{C}P^\infty)^{\otimes_{E_*} n} \to E_*(BU(n)).$$

However, since the direct sum is symmetric, this map naturally factorizes through the orbits $(E_*(\mathbb{C}P^\infty)^{\otimes n})_{\Sigma_n}$, and hence gives a map

$$\mathrm{Sym}^n_{E_*}(E_*\mathbb{C}P^\infty) = (E_*(\mathbb{C}P^\infty)^{\otimes n})_{\Sigma_n} \to E_*BU(n).$$

We let β_i be the image of $\beta_i \in E_*\mathbb{C}P^\infty$ under the standard map

$$BU(1) \to BU(n).$$

Proposition 4.25. *The homological Atiyah-Hirzebruch spectral sequences for $BU(n)$ and BU collapse at E_2. Moreover, we have that*

$$E_*BU(n) \cong \mathrm{Sym}^n_{E_*} E_*\mathbb{C}P^\infty$$

and so in the colimit we have

$$E_*BU \cong \mathrm{Sym}^\bullet_{E_*} E_*\mathbb{C}P^\infty / (\beta_0 - 1).$$

Finally, these homology groups are also coalgebras, and their coproduct is determined by

$$\psi\beta_k = \sum_{i+j=k} \beta_i \otimes \beta_j.$$

Proof. First, note that we can proceed by induction on n. Assume $n \geq 2$. Note that we have a commutative diagram

$$\begin{array}{ccc} BU(1) \times (BU(1))^{\times n-1} & \xrightarrow{1 \times \vartheta_{n-1}} & BU(1) \times BU(n-1) \\ & \searrow^{\vartheta_n} & \downarrow f \\ & & BU(n) \end{array}$$

So we can examine the map f via the Atiyah-Hirzebruch spectral sequence using the inductive hypothesis. Note that we get pairing

$$E^r(BU(1)) \otimes_{E_*} E^r(BU(n-1)) \to E^r(BU(n))$$

of Atiyah-Hirzebruch spectral sequences. If we let y_i be the class which detects β_i in the Atiyah-Hirzebruch spectral sequence for $BU(1)$, then we see that the E_2-term for $BU(n)$ is free on monomials $y_{i_1}\cdots y_{i_n}$ for $i_j \geq 0$. Since the y_i and their products are all permanent cycles, and since we have a pairing of spectral sequences, it follows that the E_2-term for $BU(n)$ is generated as an E_*-module by permanent cycles. So the AHSS collapses. This also shows that the map we produced above

$$\mathrm{Sym}^n_{E_*}(E_*\mathbb{C}P^\infty) \to E_*BU(n)$$

is in fact an isomorphism.

Since the map ϑ_n is compatible with the diagonal on both sides we find that it induces a map of coalgebras, and so we get the corresponding statement regarding the coproduct. □

One can dualize and use the Atiyah-Hirzebruch spectral sequence again (Proposition 4.19) to also show the following.

Proposition 4.26. *Since $E_*BU(n)$ is a finitely generated free module over E_*, we have that the map*

$$E^*(BU(n)) \to \hom_{E_*}(E_*BU(n), E_*)$$

is an isomorphism. We deduce that

$$E^*BU(n) \cong E^*[[c_1, \ldots, c_n]]$$

where the c_i are dual to β_1^i. Taking the colimit yields an isomorphism

$$E^*BU \cong E^*[[c_1, c_2, \ldots]].$$

Furthermore, these cohomology rings have a coproduct determined by

$$\psi c_k = \sum_{i+j=k} c_i \otimes c_j.$$

Proof. I will not give details for this. See Switzer 16.32 for details. □

At this point, we can now deduce the structure of $E_*\mathrm{MU}$. At this point, I should probably remind you about how to get a stable homological Thom isomorphism for MU. I will take as given the fact that there is a Thom isomorphism for each $BU(n)$, namely an isomorphism

$$\Phi_* : \widetilde{H}_{*+2n}(\mathrm{MU}(n);\mathbb{Z}) \to H_*(BU(n);\mathbb{Z}); x \mapsto x \frown u_{\gamma_n}$$

obtained by capping with the Thom class. It can be shown (again see Switzer) that the following diagram commutes

$$\begin{array}{ccccc}
\widetilde{H}_{*+2n}(\mathrm{MU}(n)) & \xrightarrow{\Sigma^2} & \widetilde{H}_{*+2n+2}(\Sigma^2\mathrm{MU}(n)) & \xrightarrow{M(i_n)} & \widetilde{H}_{*+2n+2}(\mathrm{MU}(n+1)) \\
\downarrow{\scriptstyle \Phi_*} & \swarrow{\scriptstyle \Phi_*} & & & \downarrow{\scriptstyle \Phi_*} \\
H_q(BU(n)) & & \xrightarrow{\qquad i_n \qquad} & & H_*(BU(n+1))
\end{array}$$

and that all of the vertical maps are isomorphisms. So in the colimit we get a "stable" Thom isomorphism

$$\Phi_* : H_*\mathrm{MU} \to H_*BU.$$

The same argument shows that whenever E is complex oriented, then we get an isomorphism

$$\Phi_* : E_*\mathrm{MU} \to E_*BU.$$

Proposition 4.27. *The map Φ_* is a ring homomorphism.*

Proof. See Lemma 16.36 of Switzer. □

We obtain as a corollary the following;

Theorem 4.28. $E_*\mathrm{MU} \cong \mathbb{Z}[b_1, b_2, \ldots]$ *where* $\Phi_*(b_i) = \beta_i$.

Recall that we have a map

$$f : \Sigma^{-2}\Sigma^{\infty}\mathrm{MU}(1) \to \mathrm{MU}.$$

We want to relate the β_i to the generators b_i above via the map f. One can show the following.

Theorem 4.29. *The map f is determined in E-homology by*

$$f_*(u^E\beta_{i+1}) = b_i^E.$$

We put in u^E so that $f_*(\beta_0) = 1$.

Proof. See Switzer □

Remark 4.30. This is taken as the definition of the b_i^E by Adams in [4]. There, Adams is implicitly using the fact that this theorem is true in integral homology. He imports this to show that if we *define* b_i^E as the class $f_*(u^E\beta_i^E)$, then this determines a basis of the E^2-page of the AHSS for MU consisting of permanent cycles. He does this so that he can completely circumvent any discussions about a Thom isomorphism for more general theories.

Theorem 4.31. *Suppose that E is a complex oriented theory with complex orientation x^E. Suppose that we have another generator of $E^*(\mathbb{C}P^\infty)$ given by*

$$f(x^E) = \sum_{i \geq 0} d_i x_E^{i+1} \in \widetilde{E}^2(\mathbb{C}P^\infty)$$

with $u^E d_0 = 1$[3]*. Then there is a unique (up to homotopy) morphism g : $\mathrm{MU} \to E$ of ring spectra*

add internal reference for footnote

$$g : \mathrm{MU} \to E$$

such that $g_* x_{\mathrm{MU}} = \varphi(x_E)$.

Proof. Since $E^*\mathrm{MU}$ and $E_*\mathrm{MU}$ are both free E_*-modules and finitely generated in each degree, we have an isomorphism

add some comments earlier about $E^*\mathrm{MU}$.

$$E^*\mathrm{MU} \to \hom_{E_*}(E_*\mathrm{MU}, E_*).$$

Similarly, we have an isomorphism

$$E^*(\mathrm{MU} \wedge \mathrm{MU}) \to \hom_{E_*}(E_*\mathrm{MU} \wedge \mathrm{MU}, E_*).$$

The first isomorphism guarantees shows that there is a bijective correspondence between homotopy classes of maps

$$g : \mathrm{MU} \to E$$

and E_*-linear maps

$$\vartheta : E_*\mathrm{MU} \to E_*.$$

The second isomorphism allows us to determine when g is a map of ring spectra, i.e. when does the following diagram commute

$$\begin{array}{ccc} \mathrm{MU} \wedge \mathrm{MU} & \xrightarrow{g \wedge g} & E \wedge E \\ \downarrow{\scriptstyle \mu^{\mathrm{MU}}} & & \downarrow{\scriptstyle \mu^{E}} \\ \mathrm{MU} & \longrightarrow & E \end{array}$$

Note that $\vartheta : E_*\mathrm{MU} \to E_*$ is a morphism of E_*-algebras if and only if the following diagram commutes,

$$\begin{array}{ccc} E_*\mathrm{MU} \otimes_{E_*} E_*\mathrm{MU} & \longrightarrow & E_* \otimes_{E_*} E_* \\ \downarrow{\scriptstyle E_*\mu^{\mathrm{MU}}} & & \downarrow \\ E_*\mathrm{MU} & \longrightarrow & E_* \end{array}$$

[3]This is a necessary condition from our previous discussions

but since the right hand vertical map is the canonical isomorphism, we can really regard this as a commutative triangle. Note that the top composite is the map corresponding to $\mu^E \circ (g \wedge g)$. Note also that there is a map

$$(E_*\mu^{\mathrm{MU}})^* : \hom_{E_*}(E_*\mathrm{MU}, E_*) \to \hom_{E_*}(E_*\mathrm{MU} \wedge \mathrm{MU}, E_*).$$

Then the second diagram commutes if and only if

$$(E_*\mu^{\mathrm{MU}})^*(\varphi) = \psi$$

and note that this latter condition is equivalent to the first diagram commuting.

The condition that

$$g_* x^{\mathrm{MU}} = \sum_{i \geq 0} d_i x_E^{i+1}$$

is equivalent to

$$\vartheta(b_i) = u^E d_i$$

for $i \geq 0$. Provided that $u^E d_0 = 1$, there is exactly one algebra map $\vartheta : E_*\mathrm{MU} \to E_*$ with this property. □

Example 4.32. As a consequence we can infer the existence of maps of ring spectra out of MU. For example, there is exactly one multiplicative map

$$H_*(\mathrm{MU}; \mathbb{Z}) = \mathbb{Z}[b_1, b_2, \ldots] \to H_*(pt; \mathbb{Z}) = \mathbb{Z}$$

which then corresponds to the map

$$g : \mathrm{MU} \to H\mathbb{Z}$$

such that

$$g_* x^{\mathrm{MU}} = x^H.$$

Likewise, there is a unique map $f : \mathrm{MU} \to KU$ such that

$$f_* x^{\mathrm{MU}} = \beta^{-1}(1 - \gamma) = x^K.$$

make this example more explicit

The map f corresponds to the map

So at this point we have a calculation of $E_*\mathrm{MU} = \pi_*(E \wedge \mathrm{MU})$ when E is complex oriented. But there is a lot more structure and things to say about this homology group in general. First, recall that the *Boardmann map* is the map

$$[X, Y] \to [X, E \wedge Y]$$

which is defined by taking $f : X \to Y$ and composing it with the map

$$Y \simeq S^0 \wedge Y \to E \wedge Y.$$

There is also the map

$$p : [X, E \wedge Y] \to \hom_{E_*}(E_*X, E_*Y); h \mapsto \langle h, - \rangle.$$

Here, $\langle h, - \rangle$ denotes the Kronecker pairing. This is defined as follows: if $h : X \to E \wedge Y$ and $k : S^0 \to E \wedge X$ then $\langle h, k \rangle$ is defined as the composite

$$S^0 \to E \wedge X \to E \wedge E \wedge Y \to E \wedge Y,$$

so $\langle h, k \rangle \in E_* Y$. We naturally have the commutative triangle

$$\begin{array}{ccc} [X,Y] & \xrightarrow{B} & [X, E \wedge Y] \\ & \searrow^{\alpha} & \downarrow p \\ & & \hom_{E_*}(E_*X, E_*Y) \end{array}$$

where α takes a map $f : X \to Y$ to $E_* f$. This triangle is interesting in the case that E is complex oriented and X or Y is $\mathbb{C}P^\infty, BU$, or MU.

So let E be a complex oriented ring spectrum. Then we have two canonical maps

$$E \simeq E \wedge S^0 \to E \wedge \mathrm{MU}$$

and

$$\mathrm{MU} \simeq S^0 \wedge \mathrm{MU} \to E \wedge \mathrm{MU}.$$

We can use these two morphisms to push-forward the generators x^E and x^{MU} to $E \wedge \mathrm{MU}$. So $E \wedge \mathrm{MU}$ has two natural complex orientations. We abuse them and call them x_E and x_{MU} to remember from whence they came. We know that these complex orientations can be related by some power series.

Lemma 4.33. *In* $(E \wedge \mathrm{MU})^*(\mathbb{C}P^\infty)$ *we have*

$$x_{\mathrm{MU}} = \sum_{i \geq 0} u_E^{-1} b_i^E x_E^{i+1}$$

where b_i^E *are the generators of* $\pi_*(E \wedge \mathrm{MU}) = E_* \mathrm{MU}$.

Proof. Let $X = \mathbb{C}P^\infty$ and $Y = \mathrm{MU}$ in the triangle above. Since x^{MU} is a reduced class, so is Bx^{MU}. Then by Theorem 4.29, we have

$$(\alpha(x^{\mathrm{MU}}))(u^E \beta_{i+1}^E) = b_i^E,$$

but we also have

$$p(x_E^j)(\beta_i^E) = \langle x_E^j, \beta_i^E \rangle = \delta_{ij}.$$

In this case, the map p is an isomorphism. So by comparing these formulas we can prove the result.

To see how, let $g(x) = \sum_{i \geq 0} d_i x^{i+1}$ such that $x_{\mathrm{MU}} = g(x_E)$. Now the class $x_{\mathrm{MU}} \in [\mathbb{C}P^\infty, \mathrm{MU}]$ is sent to $x_{\mathrm{MU}} \in (E \wedge \mathrm{MU})^* \mathbb{C}P^\infty$ under the Boardmann map. On the other hand, we also have the class $x_E \in (E \wedge \mathrm{MU})^*(\mathbb{C}P^\infty)$.

Let $h(x)$ be the formal power series

$$h(x) = \sum_{i \geq 0} u_E^{-1} b_i^E x^{i+1}.$$

We wish to show that $h(x_E) = g(x_E)$. To do that, since p is an isomorphism it is enough to show that $p(h) = p(g)$. Now, by commutativity of the diagram, we have $p(g) = \alpha(x_{\mathrm{MU}})$. Thus, from the fact that $p(x_E^j)(\beta_i^E) = \delta_{ij}$, it follows that

$$p(g)(\beta_{i+1}^E) = d_i$$

but the commutativity of the triangle shows that

$$p(g)(\beta_{i+1}^E) = u_E^{-1} b_i^E$$

Thus $g = h$ as desired. □

Corollary 4.34. *Let F_E and F_{MU} denote the formal group laws arising from the complex orientations on E and* MU *respectively, and let these also denote the induced formal group laws over $\pi_*(E \wedge \mathrm{MU})$ via the maps $E \to E \wedge \mathrm{MU}$ and* $\mathrm{MU} \to E \wedge \mathrm{MU}$. *Then*

$$F_{\mathrm{MU}}(x_1^{\mathrm{MU}}, x_2^{\mathrm{MU}}) = g(F_E(g^{-1} x_1^E, g^{-1} x_2^E))$$

where $g(x) = \sum_{i \geq 0} (u^E)^{-1} b_i^E x^{i+1}$.

4.2. **The Universal Formal Group Law and Lazard's Theorem.** At this point, I should probably say some words about the universal formal group law and how it relates to MU. Suppose that $F(x,y)$ is a formal group law over a ring R. Recall that

$$F(x,y) = x + y + \sum_{i,j>0} \alpha_{ij} x^i y^j.$$

As we mentioned before the conditions on F imply many relations on α_{ij}. Let $A := \mathbb{Z}[a_{ij} \mid i,j > 0]$ denote the polynomial ring on generators a_{ij}. Then the formal group law F uniquely determines a map

$$A \to R; a_{ij} \mapsto \alpha_{ij}.$$

However, because there are many relations amongst the α_{ij} this map has to factor through the ideal I generated by all the necessary relations amongst the α_{ij}. For example, we have $a_{ij} - a_{ji}$ in I. More generally, let

$$\widetilde{F}(x,y) := x + y + \sum_{i,j>0} a_{ij} x^i y^j$$

over A. In order to make this a formal group law, we need

$$\widetilde{F}(x,\widetilde{F}(y,z))-\widetilde{F}(\widetilde{F}(x,y),z)=\sum_{i,j,k>0} b_{ijk}x^i y^j z^k$$

is 0. So we have to add $b_{ijk}\in I$.

Definition 4.35. The *Lazard ring* is $L:=A/I$. Let F^u be the power series

$$F^u(x,y)=x+y+\sum_{i,j>0} a_{ij}x^i y^j$$

where a_{ij} is the image of $a_{ij}\in A$.

Note that, by definition, F^u is a formal group law. By construction, the following is true.

Proposition 4.36. *The functor*

$$FGL:\mathsf{Rings}\to\mathsf{Set}$$

where

$$FGL(R):=\{\textit{formal group laws over } R\}$$

is represented by L. That is, we have a natural isomorphism

$$\mathsf{Rings}(L,R)\cong FGL(R);(\varphi:L\to R)\mapsto\varphi_*F^u.$$

Remark 4.37. In light of this proposition, we call F^u the universal formal group law.

Remark 4.38. As we have seen before, if $f:R\to S$ is a morphism of rings and F is a formal group law on R, then we have a formal group law f_*F over S obtained by applying f to the coefficients of F. In the literature, one will often times see the notation f^*F instead. The rational being that f is really a map

$$f:\mathrm{Spec}(S)\to\mathrm{Spec}(R)$$

and that f is being used to *pull-back* the formal group law over to one over S.

Now, in topology, we have a grading hanging around everywhere. I have been mostly agnostic about this up to this point. But we do need to pin it down now. It also shows up in the proof of Lazard's theorem. We put a grading on L by setting $|x|=|y|=-2$ and $|a_{ij}|=2(i+j-1)$. In this way, the formal group law F^u is a homogenous expression in degree -2.

Now we don't call L the Lazard ring because it carries the universal group law, but rather because of the following difficult theorem.

Theorem 4.39 (Lazard). *As a graded ring L is given by*

$$L = \mathbb{Z}[x_1, x_2, \dots]$$

where $|x_i| = 2i$.

4.3. **Quillen's Theorem.** Quillen's theorem is the following amazing statement.

Theorem 4.40 (Quillen). *The natural map $L \to \mathrm{MU}_*$ classifying the formal group law arising from the canonical complex orientation on* MU *is an isomorphism.*

Unfortunately, I didn't really have time to prove this in class... :(

4.4. **The ring $\mathrm{MU}_*\mathrm{MU}$.** Let's now examine $\mathrm{MU}_*\mathrm{MU}$ in some further detail. From now on, I will make the tacit assumption that the complex orientations restrict to the canonical generator in $\widetilde{E}^2(\mathbb{C}P^1)$. This means we don't have any units to bother with.

First, observe that it follows from Theorem 4.28 in the case $E = \mathrm{MU}$ that

$$\mathrm{MU}_*\mathrm{MU} \cong \mathrm{MU}_*[b_1, b_2, \dots].$$

In particular, this means that the pair $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$ is a Hopf algebroid, and so we get an Adams spectral sequence based on MU. This is called the *Adams-Novikov spectral sequence*

$$\mathrm{Ext}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_*(X)) \implies \pi_* X.$$

In this case, the abutment is in fact just the homotopy groups of X. However, in order to do any sort of calculation, we really need to at least understand various formulas in the Hopf algebroid. In particular, we need to understand

$$\eta_L, \eta_R : \mathrm{MU}_* \to \mathrm{MU}_*\mathrm{MU}$$

and

$$\psi : \mathrm{MU}_*\mathrm{MU} \to \mathrm{MU}_*\mathrm{MU} \otimes_{\mathrm{MU}_*} \mathrm{MU}_*\mathrm{MU}.$$

In this case, its easier to approach this more generally and to consider $E_*\mathrm{MU}$. In this case, we have

4.5. **The Brown-Peterson spectrum.**

type up notes from these lectures... probably do over spring break

4.6. **Formulas in BP-theory.**

type up notes from these lectures... prob-
ably do over

4.7. **Formal groups.** Thus far we have been discussing formal group laws. These are certain power series with certain properties. However, they are actually attached to more geometric objects.

Fix a base ring A.

Definition 4.41. An *adic A-algebra* is an A-algebra R along with an ideal $I \subseteq R$ so that R is complete with respect to the I-adic topology. This means that

$$R \cong \lim_k R/I^k.$$

Let $adic(A)$ denote the category of adic A-algebras.

Example 4.42. The rings $A[[x_1, \ldots, x_n]]$ are examples of adic A-algebras.

Remark 4.43. Note that the functor

$$F : adic(A) \to \mathsf{Set}$$

given by sending R to the set of n-tuples of topologically nilpotent elements is represented by $A[[x_1, \ldots, x_n]]$. The functor represented by $A[[x_1, \ldots, x_n]]$ is often denoted as $\mathrm{Spf}(A[[x_1, \ldots, x_n]])$. More generally, the functor represented by R is denoted as $\mathrm{Spf}(R)$. We think of $\mathrm{Spf}(A[[x_1, \ldots, x_n]])$ as the formal affine space $\widehat{\mathbb{A}}^n$.

Definition 4.44. An *n-dimensional formal Lie variety* is a functor $F : adic(A) \to$ Set which is isomorphic to the functor represneted $A[[x_1, \ldots, x_n]]$. This gives a category of formal Lie varieties. A *formal group* is a group object in this category. Equivalently, this is representable a functor

$$F : adic(A) \to \mathrm{Ab}.$$

The relationship between 1-dimensional formal groups and formal group laws is as follows. Let $\mathbb{G}$ be a formal group over A

$$\mathbb{G} : adic(A) \to \mathrm{Ab}.$$

Then $\mathbb{G}$ is a 1-dimensional formal Lie variety, and so there is a coordinate $x : \mathbb{G} \cong \widehat{\mathbb{A}}^1$. The group operations

$$\mathbb{G} \times \mathbb{G} \to \mathbb{G}$$

then gives a diagram

$$\begin{array}{ccc} \mathbb{G} \times \mathbb{G} & \longrightarrow & \mathbb{G} \\ \big\downarrow{\scriptstyle x \times x} & & \big\downarrow{\scriptstyle x} \\ \widehat{\mathbb{A}}^1 \times \widehat{\mathbb{A}}^1 & \dashrightarrow & \widehat{\mathbb{A}}^1 \end{array}.$$

The bottom morphism is then given by a morphism

$$\varphi : A[[x]] \to A[[x]] \widehat{\otimes}_A A[[x]] \cong \mathbb{A}[[x, y]].$$

This is of course given by a formal power series $F(x,y)$, and it is easy to check that F is a formal group law.

Remark 4.45. The group axioms also show that this morphism φ is a Hopf algebra object in adic A-algebras.

In the other direction, if we have a formal group law F, then we can define a formal group $\mathbb{G}_F$ by defining the functor

$$\mathbb{G}_F : \mathrm{adic}(A) \to \mathrm{Ab}$$

by setting

$$\mathbb{G}_F(R) := \{x \in R \mid \lim_{n\to\infty} x^n = 0\}.$$

This functor takes values in abelian groups because, if $x, y \in \mathbb{G}_F(R)$, then

$$x +_F y := \lim_{n\to\infty} F_n(x,y)$$

gives an element in R. Here, $F_n(X,Y)$ denotes the polynomial obtained from F by considering only terms with degree $\leq n$.

4.8. **Height of a p-typical formal group.** Suppose F and F' are formal group laws over a ring A.

Lemma 4.46. *Let $f : F \to F'$ be a strict isomorphism of formal group laws. If $f'(0) = 0$ then $f'(x) = 0$.*

Proof. As f is an isomorphism, we have

$$fF(x,y) = F'(f(x), f(y)).$$

Applying $\frac{\partial}{\partial y}\Big|_{y=0}$ to both sides, we get

$$f'(F(x,0))F_2(x,0) = F_2'(f(x),0)f'(0).$$

Since $F(x,y) = x + y + \sum_{i,j>0} a_{ij}x^iy^j$, we see that

$$\partial_2 F(x,0) = 1 + O(x).$$

Thus $F_2(x,0)$ has a multiplicative inverse. As $f'(F(x,0)) = f'(x)$, we find

$$f'(x) = F_2(x,0)^{-1}F_2'(f(x),0)f'(0) = 0.$$

□

Suppose now that F and F' are formal group laws over a $\mathbb{F}_p$-algebra A.

Proposition 4.47. *Let F, F' be formal group laws over a $\mathbb{F}_p$-algebra A. Let $f : F \to F'$ be a strict isomorphism. Then either $f(x) = 0$ or*

$$f(x) = g(x^{p^n})$$

for some $n \geq 0$ and $g(x) \in A[[x]]$ with $g(0) = 0$ and $g'(0) \neq 0$. In particular, f has leading term x^{p^n}.

Proof. Let

$$\sigma : A \to A$$

denote the Frobenius. Then we have a formal group law $\sigma^* F$: If F is given by

$$F(x,y) = x + y + \sum_{i,j>0} a_{ij} x^i y^j$$

then

$$\sigma^* F(x,y) = x + y + \sum_{i,j} a_{ij}^p x^i y^j.$$

Note that the series $h(x) = x^p$ defines a homomorphism $h : F \to \sigma^* F$.

Now suppose that $f'(0) \neq 0$, then we take $g(x) = f(X)$. If $f(x) = 0$, then there is nothing to do.

So suppose that $f'(0) = 0$. This implies that

$$f(x) = a_p x^p + a_{2p} x^{2p} \cdots.$$

Thus, we have a factorization

$$\begin{array}{ccc} F & \xrightarrow{f} & F' \\ \downarrow{\scriptstyle h} & \nearrow{\scriptstyle g} & \\ \sigma^* F. & & \end{array}$$

Thus, $f(x) = g(x^p)$ for some power series g. We then ask if $g'(0) = 0$. If no, then we stop, otherwise we can perform the same step.

We need to check that this process stops unless $f = 0$. Let $g_n(x)$ denote the series we obtain from the nth iteration of this process. So $f(x) = g_n(x^{p^n})$. If $g_n'(0) = 0$ for all n, then it is clear that $f = 0$. □

Of particular interest is the p-series. As F is a commutative formal group, we have a homomorphism

$$[p]_F : F \to F.$$

Thus, by the above result, we have a power series $g(x)$ such that

$$[p]_F(x) = g(x^{p^n}) = v_n x^{p^n} + \cdots$$

for some $v_n \in A$ with $v_n \neq 0$.

Definition 4.48. Let $A = k$ be a field of characteristic p and let F be a formal group law over k. Then F has *height n* if

$$[p]_F(x) = v_n x^{p^n} + \cdots$$

for some $v_n \neq 0$ in k. If $[p]_F(x) = 0$ then we say that F has *height* ∞.

The point here is that if $v_n \neq 0$ in a field k, then it is a unit. If we are working more generally over an $\mathbb{F}_p$-algebra A then we need to be a bit more careful.

Definition 4.49. Let A be an $\mathbb{F}_p$-algebra and let F be a formal group law over A. Then we say that F *has height at least* n if

$$[p]_F(x) = v_n x^{p^n} + \cdots$$

with $v_n \neq 0$. We say that it *has height exactly* n if v_n is a unit in A.

Remark 4.50. Note that having height at least n doesn't change under strict isomorphism. Thus we can attach a height to any *formal group*.

The idea here is that we should regard F has giving a formal group over the base scheme Spec(A). Since A is not a field, the affine scheme Spec(A) could have many closed points. So consider a closed point

$$x : \mathrm{Spec}(\varkappa(x)) \to \mathrm{Spec}(A)$$

then we can form the pull-back $x^*\mathbb{G}_F$ of the formal group $\mathbb{G}_F$; the pullback $x^*\mathbb{G}_F$ is a formal group over $\varkappa(x)$. What can happen is that v_n could end up being 0 under the projection to the residue field at x. So the formal group $\mathbb{G}_F$ could be of higher height over some closed point x. However, if we ask that v_n is invertible in A, then v_n never projects to 0 in any residue field, and so $\mathbb{G}_F$ will *globally* have height n.

Example 4.51. If $\widehat{\mathbb{G}}_a$ is the additive formal group, then

$$[p]_{\widehat{\mathbb{G}}_a}(x) = px,$$

and so over $\mathbb{F}_p$-algebras, the p-series is identically 0.

Consider $\widehat{\mathbb{G}}_m$ the multiplicative formal group. Recall that the series was determined by

$$F(x,y) = 1-(1-x)(1-y) = x+y-xy.$$

Thus,

$$[p]_{\mathbb{G}_m}(x) = 1-(1-x)^p.$$

In the case that A is an $\mathbb{F}_p$-algebra, then the p-series becomes

$$[p]_{\mathbb{G}_m}(x) = x^p.$$

So in this case $v_1 = 1$. This shows that $\widehat{\mathbb{G}}_m$ and $\widehat{\mathbb{G}}_a$ are not isomorphic as formal groups in positive characteristic.

Now restrict to $\mathbb{Z}_{(p)}$-algebras A. Then, by Cartier's theorem, any formal group law F over A is strictly isomorphic to a p-typical one. So we may as well assume that all of our formal groups are p-typical. We can relate the Araki generators to the height.

Proposition 4.52. *Let F denote the universal p-typical formal group law over $V = \pi_* BP$ and let $v_1, v_2, \dots$ be the Araki generators. Then*

$$[p]_F(x) = px +_F v_1 x^p +_F v_2 x^{p^2} +_F \cdots .$$

The Hazewinkel generators satisfy the same equality but mod p.

Proof. We prove the case of the Araki generators. Recall that the Araki generators are recursively defined by the following equation

$$p\lambda_n = \sum_{0 \le i \le n} \lambda_i v_{n-i}^{p^i}.$$

Now note that we have

$$\log([p]_F(x)) = p\log(x) = \sum_{i \ge 0} p\lambda_i x^{p^i}.$$

Using Araki's equation, we have that the last sum can be expressed as

$$\sum_{i \ge 0} \left(\sum_{0 \le j \le i} \lambda_j v_{i-j}^{p^j} \right) x^{p^i} = \sum_{i,j \ge 0} \lambda_i v_j^{p^i} x^{p^{i+j}}.$$

Thus we have the equality

$$\log([p]_F(x)) = \sum_{i \ge 0} p\lambda_i x^{p^i} = \sum_{i,j \ge 0} \lambda_i v_j^{p^i} x^{p^{i+j}} = \sum_{j \ge 0} \log(v_j x^{p^j}).$$

Exponentiating both sides leads to

$$[p]_F(x) = \exp\left(\sum_{j \ge 0} \log(v_j x^{p^j}) \right) = \sum_{j \ge 0}^{F} v_j x^{p^j}.$$

□

Corollary 4.53. *The v_i's are integral, i.e. they are indeed elements of V.*

This is actually the first step in showing that the Araki generators are generators of V.

Theorem 4.54 (Lazard)**.** *Over a separably closed field, two formal group laws are isomorphic if and only if they have the same height.*

5. THE ALGEBRAIC CHROMATIC SPECTRAL SEQUENCE

In this section, we develop the *algebraic chromatic spectral sequence*. This is a spectral sequence which computes the E_2-term of the Adams-Novikov spectral sequence. We will begin with giving a "big picture" overview of this spectral sequence, and then describe an explicit algebraic construction.

5.1. **Big picture.** As we have discussed before, the Hopf algebroid $(L, LT) = (\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$ corresponds to formal group laws and strict isomorphisms between them, and that the Hopf algebroid $(V, VT) = (\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ corresponds to p-typical formal group laws and strict isomorphisms between those. We can be expressed in terms of the language of stacks.

If R is a ring let $\mathrm{Spec}(R)$ denote the functor represented by R, i.e.

$$\mathrm{Spec}(R) = \mathsf{CAlg}(R, -).$$

Then the pair $(\mathrm{Spec}(L), \mathrm{Spec}(LT))$ gives a functor

$$\mathsf{CAlg}_{\mathbb{Z}} \to \mathsf{Grpd}; R \mapsto (FGL(R), SI(R)),$$

where $FGL(R)$ denotes formal group laws over R and SI denotes strict isomorphisms between formal group laws over R. We can slightly rephrase this as a functor

$$\mathsf{Aff}^{op} \to \mathsf{Grpd}.$$

Here we are just using the equivalence $\mathsf{CAlg}_{\mathbb{Z}}$ with Aff^{op}.

5.2. **Algebraic construction.**

6. MORAVA'S CHANGE OF RINGS

7. THE TOPOLOGICAL STRUCTURE

APPENDIX A. STACKS AND HOPF ALGEBROIDS

In this section, I give a summary of the relationship between commutative Hopf algebroids and stacks. The main references for this appendix are [13], [16], and [7]. Another great resource for stacks is [14]. This section is not intended to give a full development of the theory, but rather just highlight the salient parts for our purposes.

A.1. **Stacks.** The starting point for thinking about stacks is the functor of points formalism of Grothendieck. Let R be a ring. Then the affine scheme $\mathrm{Spec}(R)$ can be thought of a pair consisting of the Zariski space of prime ideals of R with a sheaf of rings. A morphism of affine schemes

$$\mathrm{Spec}(R) \to \mathrm{Spec}(S)$$

is then defined as a continuous map of spaces $|f|$ along with a morphism of sheaves

$$f^{\#} : \mathcal{O}_{\mathrm{Spec}(S)} \to f^{*}\mathcal{O}_{\mathrm{Spec}(R)}.$$

Taking global sections gives a morphism of rings

$$S \to R.$$

The definitions of affine schemes are rigged up in such a way that the morphism

$$\mathrm{Spec}(R) \to \mathrm{Spec}(S)$$

and the map

$$S \to R$$

are equivalent pieces of data. Thus, we could equally well take as the definition of $\mathrm{Spec}(R)$ as the representable functor $\mathsf{CAlg}(R,-)$. The above is also saying that we have an equivalence of categories

$$\mathsf{CAlg} \simeq \mathsf{Aff}^{op}.$$

More generally, we regard a scheme X as a presheaf on affine schemes

$$X : \mathsf{Aff}^{op} \to \mathsf{Set}$$

and we intuitively think of $X(R)$ as the scheme-theoretic maps $\mathrm{Spec}(R) \to X$.

Remark A.1. In the case that $X = \mathrm{Spec}(S)$, then $X(R)$ is intuitively the scheme-theoretic maps $\mathrm{Spec}(R) \to X$, which is just a map $S \to R$. This another reason we must define $\mathrm{Spec}(S)$ as the functor $\mathsf{CAlg}(S,-)$.

In order for X to be a scheme, its not enough that this just be a functor; it must satisfy some locality conditions. For example, if we have a Zariski cover $\{U_i\}$ of $\mathrm{Spec}(R)$, then giving a map $\mathrm{Spec}(R) \to X$ ought to be the same as giving maps $U_i \to X$ which agree on overlaps. Moreover, we ask that X is Zariski locally affine.

Let k be a commutative ring. There is, of course, a variant of the above definitions for k-algebras CAlg_k. We define $\mathbb{A}^1_k := \mathrm{Spec}(k[t])$.

Definition A.2 (cf. [6])**.** Let X be a presheaf on Aff^{op}_k. The *(regular) functions* on X are the natural transformations $X \to \mathbb{A}^1_k$. This is clearly a ring and we denote it by $\mathcal{O}(X)$.

Remark A.3. Observe that for $f \in \mathcal{O}(X)$, i.e. a map $f : X \to \mathbb{A}^1_k$, we can *evaluate* f on any element $x \in X(R)$. Indeed, an element $x \in X(R)$ is just a morphism $x : \mathrm{Spec}(R) \to X$, we define $f(x)$ as the composite

$$f(x) : \mathrm{Spec}(R) \xrightarrow{x} X \xrightarrow{f} \mathbb{A}^1_k.$$

Note that $f(x) \in R$.

Definition A.4 (cf. [6]). Let X be a presheaf on Aff_k^{op} and let $E \subseteq \mathcal{O}(X)$ be a collection of regular functions on X. We define a subfunctor

$$D(E) \subseteq X$$

by declaring

$$D(E)(R) := \{x \in X(R) \mid (f(x))_{f \in E} = R\}.$$

We say that a subfunctor $Y \subseteq X$ is *open* if it is of the form $D(E)$ for some $E \subseteq \mathcal{O}(X)$.

Example A.5. Suppose that $X = \mathrm{Spec}(S)$ and that $E = \{f\} \subseteq S$. Then $D(E)(R)$ is the set of k-algebra homomorphisms $S \to R$ which sends f to a unit.

Remark A.6. Again, in the case $X = \mathrm{Spec}(R)$, then this is just $\mathsf{CAlg}_k(k[t], R)$. So in this case, the functions on X are just R.

Definition A.7 (cf. [6,7]). A presheaf of sets X on Aff_k is a *k-scheme* if it satisfies the following two conditions,

(1) X is a sheaf in the Zariski topology: if $f_1, \ldots, f_n$ are elements of a k-algebra A, and if $f_1 + \cdots + f_n = 1$, then the following

$$X(A) \longrightarrow \prod X(A[f_i^{-1}]) \rightrightarrows \prod X(A[f_i^{-1} f_j^{-1}])$$

is an equalizer diagram,

(2) X has an open cover by affine schemes.

A *morphism of schemes* is just a natural transformation of functors.

The first idea of a *stack* is to define it as a functor

$$\mathfrak{X} : \mathsf{Aff}_k^{op} \to \mathsf{Grpd}.$$

Of course, we want it to satisfy some locality conditions. Another thing, though, is that the target Grpd is not just a category, but a 2-category. So its too restrictive to ask for functors into Grpd; its better and more natural to ask that $\mathfrak{X}$ is a pseudo-functor.

Example A.8. Here is the prototypical example to illustrate why we need to consider pseudo-functors as opposed to usual 1-functors. Consider the assignment

$$\mathrm{Princ}_G : \mathsf{Top}^{op} \to \mathsf{Grpd}$$

which takes a space X to the groupoid of principal G-bundles on X, for a fixed Lie group G. Then a morphism $f : X \to Y$ of spaces induces a pull-back morphism

$$f^* : \mathrm{Princ}_G(Y) \to \mathrm{Princ}_G(X); (P \to Y) \mapsto (P \times_Y X \to X).$$

The problem is that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a composable pair of maps, then we don't have an equality of the functors $(g \circ f)^*$ and $f^* \circ g^*$, but only a natural isomorphism between the two. Thus, the above assignment doesn't define a functor, but only a pseudofunctor.

In order to avoid discussing pseudo-functors, we instead think of fibered categories. There is a correspondence between these notions going by the name of the *Grothendieck construction*. I will not spell this out, but you can look it up in [17] or [8].

Definition A.9 (cf. [13]). A *category fibered in groupoids* over $\mathcal{C}$ is a category $\mathfrak{X}$ with a functor $a : \mathfrak{X} \to \mathcal{C}$ such that the following conditions hold:

(1) For every morphism $\varphi : U \to V$ in $\mathscr{C}$ and $x \in \mathfrak{X}$ such that $a(x) = U$ there is a morphism $f : x \to y$ such that $a(f) = \varphi$,
(2) For any $f : y \to x$ in $\mathfrak{X}$ and object $z \in \mathfrak{X}$, the following is a pull-back square of sets

$$\begin{array}{ccc} \hom_{\mathfrak{X}}(z, y) & \xrightarrow{f_*} & \hom_{\mathfrak{X}}(z, x) \\ \downarrow & & \downarrow \\ \hom_{\mathscr{C}}(a(x), a(y)) & \xrightarrow{a(f)_*} & \hom_{\mathscr{C}}(a(z), a(x)) \end{array}$$

Theorem A.10 (loose statement). *A pseudofunctor $\mathscr{C}^{op} \to \mathsf{Grpd}$ is the same thing as a category $\mathfrak{X} \to \mathscr{C}$ fibered in groupoids.*

The basic idea behind this theorem is that if $\mathfrak{X} \to \mathscr{C}$ is a category fibered in groupoids, then we get a psuedofunctor $F : \mathscr{C} \to \mathsf{Grpd}$ by defining $F(V) := \mathfrak{X}_V$, the fibre of a over the object V. More precisely, $\mathfrak{X}_V$ is defined to be the subcategory of $\mathfrak{X}$ containing all objects x such that $a(x) = V$ and morphisms f such that $a(f) = 1_V$. In particular, the above theorem states that if one has morphisms $\varphi : V \to U$ in $\mathscr{C}$ and an object $x \in \mathfrak{X}_U$, then there is a unique way up to unique isomorphism to define an object $\varphi^* x \in \mathfrak{X}_V$ and a morphism $f : \varphi^* x \to x$ such that $a(f) = \varphi$.

Definition A.11. If $x \in \mathfrak{X}_U$ and $\varphi : V \to U$ is a morphism in $\mathscr{C}$, we will often write $x|V$ for $\varphi^* x$.

Exercise 23. Try proving this correspondence on your own. In particular, try to see how the conditions in the definition give you a way of defining $F(f)$ for a morphism $f : U \to V$.

Now we can define what a stack is. The basic idea is that we want it to be a "sheaf in groupoids." What this ought to mean is that morphisms between objects can be obtained by gluing them together from an open cover, and like-wise for objects. Making this precise, of course, requires effort.

We need to assume that $\mathscr{C}$ has finite limits.

Definition A.12. Let $\mathscr{C}$ be a category with a Grothendieck topology. A *stack* over $\mathscr{C}$ is a category fibered in groupoids $\mathfrak{X}$ over $\mathscr{C}$ such that

(1) (Descent for morphisms) Given an object $U \in \mathscr{C}$ and objects $x, y \in \mathfrak{X}_U$, the functor

$$\mathscr{C}_{/U}^{op} \to \mathsf{Set}; (\varphi : V \to U) \mapsto \hom_{\mathfrak{X}_V}(\varphi^* x, \varphi^* y)$$

is a sheaf of sets,

(2) (Descent for objects) Given any cover $\{U_i \to U\}$ of U in the Grothendieck topology, objects $x_i \in \mathfrak{X}_{U_i}$ and isomorphisms

$$\tau_{ij} : x_i | U_i \times_U U_j \to x_j | U_i \times_U U_j$$

which satisfy the cocycle condition, then there is an object $x \in \mathfrak{X}_U$ and isomorphisms $f_i : x|U_i \to x_i$ such that $f_j | U_i \times_U U_j = \tau_{ij} \circ f_i | U_i \times_U U_j$.

Example A.13. Let $\mathscr{C} = \mathsf{Top}$ with the usual topology and let $\mathfrak{X} = \mathrm{Princ}_G$ be the category whose objects are principal G-bundles $P \to X$ over any base space X and with the obvious morphisms. This defines a stack over Top. Analogous examples can be replaced when $\mathscr{C} = \mathrm{Sch}_k$ the category of k-schemes and $\mathfrak{X}$ is the category of vector bundles on schemes, or quasi-coherent sheaves on schemes, etc. In the case of algebraic geometry, there are many different topologies to choose from, such as the $fpqc$, $fppf$, étale, Nisnevich, Zariski, etc.

Remark A.14. While I have phrased the above definition in a very general fashion, for us we will always take $\mathscr{C}$ to be Aff, and we do so from this point onward. For concreteness we also endow Aff with the fpqc topology.

Example A.15. An important example for us is $\mathscr{M}_{fg} \to \mathsf{Aff}$. In this case $\mathscr{M}_{fg}$ is the category whose objects are pairs $(\mathrm{Spec}(R), \mathbf{G})$ where $\mathbf{G}$ is a formal group over $\mathrm{Spec}(R)$. A morphism of pairs is

$$(f, \varphi) : (\mathrm{Spec}(R), \mathbf{G}) \to (\mathrm{Spec}(T), \mathbf{G}')$$

where $f : T \to R$ is a map of rings and $\varphi : \mathbf{G} \to f^* \mathbf{G}'$ is a (strict) isomorphism of formal groups.

Now, keep in mind that the category of categories fibered in groupoids over a fixed base category $\mathscr{C}$ is a 2-category:

Objects Objects are categories $a : \mathfrak{X} \to \mathscr{C}$ fibered in groupoids,

1-morphisms 1-morphisms are functors $F : \mathfrak{X} \to \mathfrak{Y}$ which make the following diagram commute:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{F} & \mathfrak{Y} \\ & \searrow \quad \swarrow & \\ & \mathscr{C} & \end{array}$$

2-morphisms natural isomorphisms between functors

Definition A.16. A 1-morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of stacks is *representable* if for any scheme U with a 1-morphism $X \to \mathfrak{Y}$, the fibre product $\mathfrak{X} \times_{\mathfrak{Y}} X$ is a scheme.

Remark A.17. In the literature, people often ask instead that the fibre product $\mathfrak{X} \times_{\mathfrak{Y}} X$ is an algebraic space.

In general, if P is some property of schemes (or algebraic spaces) and $f : \mathfrak{X} \to \mathfrak{Y}$ is a representable 1-morphism of stacks, then we say that f has property P if whenever, U is a scheme and $U \to \mathfrak{Y}$ is a 1-morphism, then the map

$$\mathfrak{X} \times_{\mathfrak{Y}} U \to U$$

is a morphism of schemes with property P. Note that in order for the fibre product $\mathfrak{X} \times_{\mathfrak{Y}} U$ to be a scheme we need to assume that $f : \mathfrak{X} \to \mathfrak{Y}$ is a representable morphism.

Definition A.18 (cf. [7, 13])**.** A stack $\mathfrak{X} \to \mathsf{Aff}$ is *algebraic* if the diagonal 1-morphism $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is a representable morphism and there is an affine scheme U and a faithfully flat 1-morphism $P : U \to \mathfrak{X}$. We call P a *presentation* of $\mathfrak{X}$.

Remark A.19. Recall that a morphism is faithfully flat if and only if it is flat and surjective.

Remark A.20 (cf. [14])**.** The condition that the diagonal 1-morphism is representable implies that for any scheme X and morphism $f : X \to \mathfrak{X}$ is representable. Indeed, suppose that $u : U \to \mathfrak{X}$ is a morphism from a scheme U into $\mathfrak{X}$. Then the fibre product $U \times_{u,\mathfrak{X},t} T$ is also given as the pull-back of the following diagram

$$\begin{array}{ccc} & & U \times T \\ & & \big\downarrow{\scriptstyle u\times t} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array}.$$

This is what allows us to even ask that the morphism P is faithfully flat.

Remark A.21. The definition of an algebraic stack above is the one homotopy theorists tend to use. It is different from what is common in the algebraic geometry literature. The algebraic geometers tend to ask that P is a smooth surjective morphism. Asking that P is a smooth map implies that it is flat and locally of finite type. In homotopy theory, however, the main example is $P : \mathrm{Spec}(L) \to \mathscr{M}_{fg}$, but this is not locally of finite type, and hence not smooth. This can be seen as follows. We have the fibre product

$$\begin{array}{ccc} \mathrm{Spec}(L) \times_{\mathscr{M}_{fg}} \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(L) \\ \big\downarrow & & \big\downarrow{\scriptstyle P} \\ \mathrm{Spec}(L) & \xrightarrow{P} & \mathscr{M}_{fg} \end{array}$$

and the fibre product is $\mathrm{Spec}(LT)$. Clearly, $\mathrm{Spec}(LT) \to \mathrm{Spec}(L)$ is not locally of finite type since LT is not a finitely generated L-algebra.

Definition A.22. A *rigidified algebraic stack* is a pair $(\mathfrak{X}, P : U \to \mathfrak{X})$ where $\mathfrak{X}$ is an algebraic stack such that the diagonal is affine and P is a presentation of $\mathfrak{X}$.

Remark A.23. That the diagonal is affine is part of the definition of an algebraic stack in [13], but this is not required in other places in the literature.

A.2. **Flat Hopf algebroids and stacks.** Suppose that (A, Γ) is a flat Hopf algebroid. Then, the pair of functors

$$(\mathrm{Spec}(A), \mathrm{Spec}(\Gamma)) : \mathsf{Aff}^{op} \to \mathsf{Grpd}$$

determines a functor into groupoids. It is also convenient to think of $(\mathrm{Spec}(A), \mathrm{Spec}(\Gamma))$ as a groupoid itself by regarding $\mathrm{Spec}(A)$ as the affine scheme of objects and $\mathrm{Spec}(\Gamma)$ as the affine scheme of morphisms.

Now suppose that (B, Φ) is another flat Hopf algebroid. Then a functor of groupoids

$$F : \mathrm{Spec}(A), \mathrm{Spec}(\Gamma) \to \mathrm{Spec}(B), \mathrm{Spec}(\Phi)$$

corresponds, by Yoneda, to a morphism

$$f_0 : B \to A$$

and

$$f_1 : \Phi \to \Gamma$$

such that the obvious diagrams commute. For example, the fact that a functor commutes with composition implies the following diagram commutes

$$\begin{array}{ccc} \Phi & \xrightarrow{\quad g \quad} & \Gamma \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\ \Phi \otimes_B \Phi & \longrightarrow & \Gamma \otimes_A \Gamma. \end{array}$$

Exercise 24. Figure out all of the necessary commutative diagrams.

However, the category of groupoids is actually a 2-category. So the category of flat Hopf algebroids ought to be a 2-category as well. The 2-morphisms should correspond to natural transformations of functors. Suppose that

$$F, G : (\mathrm{Spec}(A), \mathrm{Spec}(\Gamma)) \to (\mathrm{Spec}(B), \mathrm{Spec}(\Phi))$$

are functors between groupoids. Then a natural transformation $\eta : F \to G$ is given by a morphism of affine schemes

$$\eta : \mathrm{Spec}(A) \to \mathrm{Spec}(\Phi),$$

which picks out for every object x a morphism $\eta_x : Fx \to Gx$. By Yoneda, this is just a map $c : \Phi \to A$.

This latter part suggests further conditions on η. Namely, we ought to have that $\eta_L \eta = f_0$ and $\eta_R \eta = g_0$. Moreover, being a natural transformation requires that certain diagrams commute. This can be expressed in terms of these affine schemes that the following diagram

$$\begin{array}{ccc} \mathrm{Spec}(\Gamma) & \xrightarrow{\quad (g_1\, \eta s) \quad} & \mathrm{Spec}(\Phi) \times_{s, \mathrm{Spec}(B), t} \mathrm{Spec}(\Phi) \\ \downarrow{\scriptstyle (\eta t\, f_1)} & & \downarrow{\scriptstyle \Delta} \\ \mathrm{Spec}(\Phi) \times_{s, \mathrm{Spec}(B), t} \mathrm{Spec}(\Phi) & \xrightarrow{\quad \Delta \quad} & \mathrm{Spec}(\Phi) \end{array}$$

Clearly, if $\eta : F \to G$ and $\eta' : G \to H$ are two natural transformations, then we can compose them componentwise. On schemes, the composite $\eta' \circ \eta$ is given by

$$\eta' \circ \eta : \mathrm{Spec}(A) \xrightarrow{(\eta'\, \eta)} \mathrm{Spec}(\Phi) \times_{\mathrm{Spec}(B)} \mathrm{Spec}(\Phi) \xrightarrow{\Delta} \mathrm{Spec}(\Phi).$$

We can now describe a functor from rigidified algebraic stacks and flat Hopf algebroids.

Definition A.24. Let $\mathsf{RigStack}$ denote the 2-category of rigidified algebraic stacks and let $\mathsf{HopfAlgd}^\flat$ denote the category of flat Hopf algebroids.

We can define a functor

$$K : \mathsf{RigStack} \to \mathsf{HopfAlgd}^\flat$$

as follows. Let $(\mathfrak{X}, P : X_0 \to \mathfrak{X})$ be a rigidfied algebraic stack. Then we can form the fibre product $X_1 := X_0 \times_{P,\mathfrak{X},P} X_0$. We require that X_0 is affine, and this implies that the fibre product X_1 is also affine. Furthermore, it is clear that (X_0, X_1) naturally has the structure of a groupoid.

There are maps

$$s, t : X_1 \to X_0$$

coming from projecting onto the first or second factor, and there is a map

$$\chi : X_1 \to X_1$$

arising from switching the factors. Since the presentation P is faithfully flat, it follows that the maps s, t are flat maps of affine schemes. In other words, (X_0, X_1) is a flat Hopf algebroid. Thus K is really a functor into flat Hopf algebroids.

Exercise 25. Examine how K behaves on 1-morphisms and 2-morphisms and show that K is in fact a 2-functor $\mathsf{RigStack} \to \mathsf{HopfAlgd}^\flat$.

There is a two functor going in the opposite direction,

$$G : \mathsf{HopfAlgd}^\flat \to \mathsf{RigStack},$$

but it is a bit more complicated to write down.

First, suppose that (X_0, X_1) is a flat Hopf algebroid (we think of X_0 and X_1 as affine schemes). Then we can define a stack $\mathfrak{X}$ associated with (X_0, X_1) as follows. First, recall that we can consider this pair as a functor

$$(X_0, X_1) : \mathsf{Aff}^{op} \to \mathsf{Grpd},$$

we can apply the Grothendieck construction to get a category fibered in groupoids over Aff, call this category $\widetilde{\mathfrak{X}}_{(X_0,X_1)}$. This may not be a stack, but we can "stackify" it. This is the stack $\mathscr{M}_{(X_0,X_1)}$ with a canonical map $\widetilde{\mathfrak{X}}_{(X_0,X_1)} \to \mathscr{M}_{(X_0,X_1)}$ which has the property that a map

$$\widetilde{\mathfrak{X}}_{(X_0,X_1)} \to \mathfrak{Y}$$

of categories fibered in groupoids over Aff with $\mathfrak{Y}$ a stack uniquely factors through $\mathscr{M}_{(X_0,X_1)}$. By construction this stack will be an algebraic stack and its diagonal will be affine. It also receives a map from X_0, $P : X_0 \to \mathscr{M}_{(X_0,X_1)}$. This map turns out to be representable and faithfully flat since $s, t : X_1 \to X_0$ are faithfully flat.

A.3. Quasi-coherent sheaves and comodules.

A.4. Cohomology and Ext.

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