MATH 402 Worksheet 6

Friday 3/8/18

Exercise 1. Let f be an isometry of the plane. Show that f preserves tangent lines to circles and tangent circles. Show that f also preserves orthogonal circles.

Exercise 2.

(1) Recall that a translation τ is an isometry which is a composite $r_m \circ r_\ell$ where m and ℓ are parallel or coincident lines. Show that if a translation τ is not the identity, then τ fixes no points.
(2) Let P and Q be distinct points. Show that there is a translation $\tau_{\overrightarrow{PQ}}$ which takes P to Q.
(3) Show that the composite of two translations is again a translation.
(4) Show that the inverse of a translation is again a translation.
(5) Suppose that f is an isometry which has no fixed points. Show that f can be written as a composition $R \circ \tau$ where τ is a translation and R has at least one fixed point.

Exercise 3.

(1) Let $E(2)$ denote the group of isometries of the plane (identified with the Cartesian plane $\mathbb{R}^2$). Let $O(2)$ denote the isometries which fix the origin, and let $T(2)$ denote the translations (including the identity). Show that $O(2)$ and $T(2)$ are subgroups of $E(2)$ (this just means the composite of two isometries in $O(2)$ or $T(2)$ are elements of $O(2)$ or $T(2)$ respectively, and the inverse of any element is again in $O(2)$ or $T(2)$ respectively.)
(2) Show that these subgroups generate $E(2)$. This means that any isometry is the composite of an element of $O(2)$ and an element of $T(2)$.
(3) A subgroup H of a group G is *normal* if for any $x \in H$ and $g \in G$, one has $g^{-1}xg \in H$. Show that $O(2)$ is a normal subgroup of $E(2)$.
(4) One can use the other parts of this exercise to show that the group of isometries of $\mathbb{R}^2$ is the semi-direct product $O(2) \rtimes T(2)$. (If you don't know what this is, don't worry. This last part is not requried.)