MATH 402 Worksheet 7

Friday 4/6/18

Exercise 1. Recall that we defined in class what it meant for a ray Aa to be a *limiting parallel* of a ray Bb, which we will denote by $Aa|||Bb$. In this exercise, you will sketch out the proof that this defines an equivalence relation.

(1) Show that the relation is symmetric, i.e. if $Aa|||Bb$ then $Bb|||Aa$. You will need to rotate and use Pasch's axiom to make this argument. (Drawing a picture may be helpful)
(2) Show that this relation is transitive, i.e. if $Aa|||Bb$ and $Bb|||Cc$ then $Aa|||Cc$. To prove this, first show that if a, b, c are distinct lines, then we can make A, B and C to be collinear.

Since the relation of being a limiting parallel defines an equivalence relation, we can make the following definition

Definition 1. An *omega point* is an equivalence class of rays under the relation of being limiting parallels. A *generalized point* will refer to either a point in the usual sense or an omega point.

Thus, given a line ℓ, and points P, Q on ℓ, there is an omega point Ω determined by the ray $\overrightarrow{PQ}$.

Exercise 2.

(1) Show that any line ℓ determines two distinct omega points. That there are two arises from the fact that, once a point P is fixed on ℓ, there are two sides of ℓ (this is what Hvidsten confusingly refers to as left and right).
(2) We will say that two lines ℓ and m intersect at an omega point if they share a common omega point. Give a more rigorous mathematical definition of this concept.
(3) Show that lines ℓ and m intersect at an omega point if and only if ℓ is a limiting parallel of m.
(4) Show that given any point P and any omega point ω, there is a unique line ℓ passing through P and ω.
(5) Given concrete descriptions of omega points for the Klein and Poincaré disc models.
(6) Draw pictures describing triangles where
    (a) one of the vertices is an omega point
    (b) two of the vertices are omega points
    (c) all of the vertices are omega points
(7) Let $Aa|||Bb$ and $A'a'|||B'b'$. Suppose that $\angle BAa \cong \angle B'A'a'$ and that $AB \cong A'B'$. Show that $\angle ABb \cong \angle A'B'b'$.
(8) Given four rays $Aa, Bb, A'a'$ and $B'b'$, suppose that $\angle BAa \cong \angle B'A'a'$ and $\angle ABb \cong \angle A'B'b'$ and that $AB \cong A'B'$. Then $Aa|||Bb$ if and only if $A'a'|||B'b'$.

Definition 2. The *extended hyperbolic plane* will consist of the points of the hyperbolic plane and the omega points of the plane.

Exercise 3. Let f be an isometry of the hyperbolic plane. Show that f naturally extends to the extended hyperbolic plane.