WHAT ARE SPECTRA?

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Contents

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1. Introduction

These notes are growing out of my desire to completely relearn the foundations of stable homotopy theory, and I have decided to make them public for the class. I will try to update them periodically. I claim no originality in this material, except possibly the mistakes.

Disclaimer: At the moment, these notes are *extremely unorganized* and are subject to change on a nearly daily basis. Use with caution.

2. What is the stable homotopy category supposed to be?

Suppose we have two finite pointed CW complexes X and Y, and suppose we have pointed maps $f, g : \Sigma^r X \to Y$. How can we tell whether or not f and g are homotopic or not? One way is to find a cohomology theory E^* which sees the difference between f and g: that is, we want $E*(f) \neq E^*(g)$. If we can find such an E, then we can distinguish between f and g.

But this begs the question, have we really distinguished between the *homotopy classes* of f and g? Well, not quite. As we have discussed in class, (co)homology theories are stable under suspension, which means any suspension of f and g will have the same effect in cohomology. However, the following scenario could occur. It may happen that $f, g : \Sigma^r X \to Y$ are not homotopic, but are *stably homotopic*, that is, there is some $\ell \gg 0$ such that $\Sigma^\ell f$ and $\Sigma^\ell g$ are homotopic. If this is the case then E^* cannot tell the difference between f and g, despite the fact that f and g are not homotopic. What E^* can distinguish are between *stable homotopy classes* of maps.

Example 2.1. Consider the fibration

$$S^1 \to S^3 \to S^2$$

arising out of the usual definition of $\mathbb{C}P^1$. Then we get a long exact sequence in homotopy, which yields $\pi_3 S^2 = \mathbb{Z}\{\eta\}$, where η is the so-called first Hopf invariant one element. In particular, 2η is not nullhomotopic. However, it is known that $2\Sigma\eta$ is null homotopic. So 2η is not null homotopic but it is stably nullhomotopic.

Thus, if we are to try to understand the category of spaces through the eyes of cohomology theories, it is natural to find a category which incorporates all of the identifications and properties seen by *all* cohomology theories. That is, we want a category SHC such that for any cohomology theory E, we have the following factorization

$$\begin{array}{ccc} \mathsf{ho}\,\mathsf{Top}_* & \xrightarrow{\widetilde{E}^*} & \mathcal{A} \\ \Big\downarrow{\scriptstyle \Sigma^\infty} & \nearrow{\scriptstyle \exists !} & \\ \mathsf{SHC} & & \end{array}$$

where $\mathcal{A}$ is any kind of suitable category (probably pointed and additive). We call this desired category SHC the *stable homotopy category*. I have suggestively written down some sort of functor

$$\Sigma^\infty : \mathsf{ho}\,\mathsf{Top}_* \to \mathsf{SHC},$$

we should think of this functor as taking a space X to its "universal stable invariant" $\Sigma^\infty X$. That is, any sort of stable invariant we can extract out of X is really just the same as an invariant of $\Sigma^\infty X$. The following prospect then presents itself: to study stable invariants of X, its best to study the topology of the object $\Sigma^\infty X$. Properties of stable invariants of X will then be consequences of the properties of $\Sigma^\infty X$.

Now there are many properties we might expect the category SHC to have. For one, we would like there to be a right adjoint to the functor Σ^∞:

$$\Sigma^\infty : \mathsf{ho}\,\mathsf{Top}_* \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} : \mathsf{SHC}.$$

Furthermore, we want SHC to capture *stable* homotopy theory. Since the suspension functor is seen as an invertible operation in (co)homology theories, this suggests that we want

- The category SHC to have a suspension functor $\Sigma : \mathsf{SHC} \to \mathsf{SHC}$ which is compatible with the usual suspension on pointed spaces, i.e. the following diagram commutes

$$\begin{array}{ccc} \mathsf{ho}\,\mathsf{Top}_* & \xrightarrow{\Sigma} & \mathsf{ho}\,\mathsf{Top}_* \\ \Big\downarrow{\scriptstyle \Sigma^\infty} & & \Big\downarrow{\scriptstyle \Sigma^\infty} \\ \mathsf{SHC} & \xrightarrow{\Sigma^\infty} & \mathsf{SHC} \end{array}$$

- Since (co)homology theories see Σ as an invertible operation, we want $\Sigma : \mathsf{SHC} \to \mathsf{SHC}$ to be an equivalence of of SHC.

Furthermore, we found in $\mathsf{ho}\,\mathsf{Top}_*$ that fibre and cofibre sequences were very useful tools, we want an analogue of sequences in spectra. Moreover, we want to have cofibre sequences mapped to cofibre sequences under Σ^∞.

Also, we want for based spaces X and Y the following formula

$$[\Sigma^\infty X, \Sigma^\infty Y] \cong \operatorname*{colim}_{k\to\infty}[\Sigma^k X, \Sigma^k Y]$$

In particular, we must have that the hom sets in SHC between suspension spectra are abelian groups. We want this to be true for any hom set between any two spectra. Thus, we want SHC to be enriched in abelian groups. This will have two very interesting consequences:

- The category SHC has products and coproducts given by $\times$ and $\vee$ respectively. Namely, for X and Y spectra, we have an equivalence $X \vee Y \to X \times Y$.
- If $f : A \to X$ is a map of spectra and $r : X \to A$ is a retract of f, i.e. $rf = 1$, then there is a splitting $X = A \vee \Omega C i$.

Neither of these two things happened in spaces. These properties hold true for any pointed additive category.

Finally, we really want there to be a functor $\wedge$ on SHC which makes this into a symmetric monoidal category. We also want a functor $F(-,-)$ which makes SHC into a closed symmetric monoidal category, i.e. so that

$$[X \wedge Y, Z] \cong [X, F(Y, Z)].$$

I should also mention, we want SHC to be the homotopy category of some category. This category is often called Sp for the category of spectra. We would like this category to have the following properties: For a space X let

$$QX := \operatorname{colim}\left(X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \cdots\right)$$

where $X \to \Omega\Sigma X$ is the unit of the adjunction $\Sigma \dashv \Omega$. We want the category Sp to have functors

$$\Sigma^\infty : \mathsf{Top}_* \to \mathsf{Sp}$$

and

$$\Omega^\infty : \mathsf{Sp} \to \mathsf{Top}_*$$

We want these to satisfy

A1 The category Sp is symmetric monoidal with respect to the smash product of spectra,

A2 The functor Σ^∞ is left adjoint to the functor Ω^∞,

A3 The unit for the smash product is $\Sigma^\infty S^0$, A4 Either there is a natural transformation

$$\varphi : \Omega^\infty D \wedge \Omega^\infty E \to \Omega^\infty(D \wedge E)$$

or

$$\psi : \Sigma^\infty(X \wedge Y) \to \Sigma^\infty X \wedge \Sigma^\infty Y$$

which commute with the monoidal structures. (i.e. they are lax monoidal functors.)

A5 There is a natural weak equivalence

$$\vartheta : \Omega^\infty \Sigma^\infty X \to QX$$

for $X \in \mathsf{Top}_*$ which makes the following diagram commute

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \Omega^\infty\Sigma^\infty X \\ & \searrow^{\iota} & \downarrow{\scriptstyle\vartheta} \\ & & QX \end{array}$$

Theorem 2.2 (Lewis)**.** *There is no such category of spectra satisfying (A1)-(A5).*

3. The Adams-Boardmann Category

In this section, we present what is probably the most classical category of spectra. This is the category of spectra which can be found in Adams' Blue book or in Switzer.

3.1. **The category.** Let's first define the category, which is to say, the objects and the morphisms. Along the way, we will also define what its homotopy category is.

Definition 3.1. A *CW-spectrum* E is a sequence of CW complexes $E = \{E_n\}_{n\in\mathbb{Z}}$ along with inclusions $\Sigma E_n \to E_{n+1}$ which exhibits ΣE_n as a subcomplex of E_{n+1}.

We can define the homotopy groups of a CW spectrum.

Definition 3.2. Let $r \in \mathbb{Z}$, define

$$\pi_r E = \operatorname*{colim}_n \pi_{n+r}(E_n).$$

Here the colimit is over the diagram where the map $\pi_{n+r}E_n \to \pi_{n+r+1}E_{n+1}$ is

$$\pi_{n+r}E_n \xrightarrow{\Sigma} \pi_{n+r+1}(\Sigma E_n) \longrightarrow \pi_{n+r+1}(E_{n+1}).$$

Remark 3.3. Of course, we only start the diagram when $n + r \geq 0$. Thus if $r < 0$, we require that $n \geq -r$.

Example 3.4. If E is a spectrum, then $\pi_{-1}E$ is the colimit of

$$\pi_0(E_1) \to \pi_1 E_2 \to \pi_2(E_3) \to \cdots$$

Definition 3.5. Let $f : E \to F$ be a function of spectra. We call f a *stable equivalence* if $\pi_* f : \pi_* E \to \pi_* F$ is an isomorphism on homotopy groups.

In classical topology, CW complexes have cells, we want to say that CW-spectra also have cells. Suppose that e_n^d is a d-dimensional cell of the complex E_n. Then suspending embeds ΣE_n in E_{n+1} as a subcomplex. So, we see that Σe_n^d is a $(d+1)$-dimensional cell in E_{n+1}. Inductively, we find that $\Sigma^k e_n^d$ is a $(d+k)$-dimensional cell in E_{n+k}. Now, it may happen that the cell e_n^d actually originated at some previous stage. That is to say, there is cell $e_{n'}^{d'}$ in $E_{n'}$ which suspends to e_n^d. For this to be the case, we would have to have

$$\Sigma^{d-d'} e_{n'}^{d'} = e_n^d.$$

In this case, we would have $n = n' + d - d'$, and hence $n' = n + d' - d$. Since $d - d' \geq 0$, the cell e_n^d cannot be "desuspended" infinitely many times. Rather, it can only be desuspended at most $d - d'$ many times, and so there is a *first* cell which gives rise to e_n^d (of course it might happen that e_n^d is the first). We call the collection

$$e = \{e_{n'}^{d'}, \Sigma e_{n'}^{d'}, \ldots\}$$

a *stable cell of dimension* $d - n$ of the CW spectrum E. We say a spectrum is *finite* if there are only finitely many stable cells.

Exercise 1. Show that E is a CW spectrum with only finitely many stable cells if and only if there is an integer N such that E_N is a finite complex and for all $n \geq N$ we have $E_n = \Sigma^{n-N} E_N$.

Remark 3.6. Here is how we got the dimension. Observe that for any of the cells in the collection e, the difference between the dimension of the cells and the space they appear in the spectrum is $d - n$, which is constant for each element of the family.

Remark 3.7. Later, we will talk about desuspensions, and we will see that a spectrum E can be written as a colimit

$$E = \operatorname*{colim}_n \Sigma^{-n} E_n,$$

this coincides with how we got stable dimensions earlier.

Here is a very important collection of examples of CW spectra.

Definition 3.8. Let X be a CW-complex. Define $\Sigma^\infty X$ to be the CW-spectrum defined by

$$(\Sigma^\infty X)_n := \Sigma^n X$$

and morphisms

$$\Sigma\Sigma^n X \to \Sigma^{n+1} X$$

the identity map.

Exercise 2. Show that for $n \geq 0$ we have $\pi_n(\Sigma^\infty X) = \pi_n^{st}(X)$. Show that for $n < 0$, one has $\pi_n \Sigma^\infty X = 0$.

Of course, we want a category of spectra, and so we ought to define what morphisms are. The most obvious definition is

Definition 3.9. A *function* $f : E \to F$ between CW-spectra is a sequence of cellular maps $f_n : E_n \to F_n$ so that the following diagram commutes

$$\begin{array}{ccc} \Sigma E_n & \longrightarrow & E_{n+1} \\ \downarrow{\scriptstyle \Sigma f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma F_n & \longrightarrow & F_{n+1} \end{array}$$

Exercise 3. Show that a function $f : E \to F$ between spectra induces a map on homotopy groups.

Definition 3.10. A function which induces an isomorphism on homotopy groups is called a *stable equivalence*.

There is another notion of equivalence that one may have come up with.

Definition 3.11. A *levelwise equivalence* is a function $f : E \to F$ between spectra so that each component $f_n : E_n \to F_n$ is a weak equivalence.

Exercise 4. Show that levelwise equivalences induce isomorphisms on homotopy groups, and so are stable equivalences.

For various reasons, functions are not sufficient for our purposes.

Exercise 5. Show that for CW complexes X and Y, the functions $f : \Sigma^\infty X \to \Sigma^\infty Y$ are in one-to-one correspondence with the set of cellular maps $X \to Y$.

This exercise shows that the notion of a function is really no good. Indeed, consider the first Hopf invariant one element $\eta : S^3 \to S^2$. This determines an element in $\pi_1^{st}(S^0)$. We would like this element to correspond to a morphism $\eta : \Sigma^\infty S^1 \to \Sigma^\infty S^0$ in our category. However, by the previous exercise, if this were a funtion, then it is uniquely determined by a map $S^1 \to S^0$. But there are no maps $S^1 \to S^0$ or $S^2 \to S^1$ which suspend to η. Here is another example,

Example 3.12. Take the two spectra E and F with

$$E_n = \bigvee_{k\geq 1} S^{n+4k-1}$$

and

$$F_n = S^n.$$

We would like to have a function $E \to F$ whose component from S^{n+4k-1} picks up the generator of the stable image of J in stem $4k-1$. However, there is no single value of n for which all the required maps exist.

This example shows that we really must come up with a different notion of morphism in our category.

Definition 3.13. Let E be a CW spectrum. A *subspectrum* of E is a sequence of subcomplexes $F_n \subseteq E_n$ so that ΣF_n is a subcomplex of F_{n+1}. We say that F is *cofinal* if every cell of E contains an element which is a cell of some F_n.

Remark 3.14. Note that if X is a CW spectrum and A is a subspectrum, then we can define the relative homotopy groups $\pi_*(X,A)$. Indeed, they are are just given as the colimit

$$\pi_r(X,A) = \operatorname*{colim}_n \pi_{n+r}(X_n, A_n).$$

The long exact sequence of a pair in homotopy (and taking a colimit) gives a long exact sequence

$$\cdots \to \pi_r A \to \pi_r X \to \pi_r(X,A) \to \pi_{r-1}(A) \to \cdots.$$

for pairs of spectra.

Definition 3.15. A *filtration* of CW spectrum E is an increasing sequence of CW spectra $\{E^n \mid n \in \mathbb{Z}\}$ whose union is all of E.

Example 3.16. Let $E^{(n)}$ denote the subspectrum which is the union of all cells of dimension less than or equal to n. In particular $E^{(n)}_m \subseteq E_m$ is the $(n+m)$-skeleton of E_m. This is known as *skeletal filtration*. Unlike in the category of spaces, a spectrum E can have have cells in arbitrarily negative dimension. This makes the skeletal filtration useless for doing inductive arguments, in stark contrast to the category of spaces.

Here is a more convenient description of a cofinal subspectrum.

Proposition 3.17. *Let E be a CW spectrum. Then a subspectrum F is a cofinal if and only if for every n and finite subcomplex $K \subseteq E_n$, there is a d such that $\Sigma^d K \subseteq F_{n+d}$.*

Cofinal subspectra have nice closure properties.

Proposition 3.18. *The intersection of two cofinal subspectra is cofinal. Furthermore, if $G \subseteq F \subseteq E$ such that F is cofinal in E and G is cofinal in F then G is cofinal in E. Arbitrary unions of cofinal subspectra are cofinal.*

Definition 3.19. Let E and F be CW spectra. Consider the set of all pairs (E', f) where $E' \subseteq E$ is a cofinal subspectrum and $f : E' \to F$ is a function of CW spectra. Define an equivalence relation on this set by regarding (E', f') equivalent to (E'', f'') if there is a cofinal subspectrum $E''' \subseteq E' \cap E''$ such that $f'|_{E'''} = f''|_{E'''}$. A *map* $E \to F$ is defined to be an equivalence class of such pairs.

What is this intuitively saying. Well, its saying that if you have a cell e in the spectrum E, then a *map* will not necessarily be defined on the cell e, but it will be defined on *some* suspension of e. The slogan we get from Adams is "cells now, maps later."

We would like to define composition on maps, so that we get a category. The problem is that if we have a representative (E', f') and (F', g') we may not be able to compose g' and f', since it may not happen that $f'(E') \subseteq F'$. We need a lemma.

Lemma 3.20. *Let $f : E \to F$ be a function of CW spectra, and let $F' \subseteq F$ be a cofinal subspectrum. Then there is a cofinal subspectrum $E' \subseteq E$ such that $f(E') \subseteq F'$.*

Proof. Consider the collection of all subspectra G such that $f(G) \subseteq F'$. This collection is nonempty because it contains the basepoint. Let E' be the union of these subspectra. Then $f(E') \subseteq F'$. It remains to show that E' is a cofinal subspectrum. Let K be a finite complex in E_n. Consider $f_n(K)$, this is contained in a finite subcomplex $H \subseteq F_n$. This is because f_n is cellular. As F' is cofinal, there is a d such that $\Sigma^d H \subseteq F'_{n+d}$. Thus $f_{n+d}(\Sigma^d K) \subseteq F'_{n+d}$. So $\Sigma^d K \subseteq E'_{n+d}$. □

This allows us to compose maps. Indeed, suppose we have a map $(E', f') : E \to F$ and a map $(F', g') : F \to G$. Then, by the lemma, there is a cofinal subspectrum E'' such that $f'(E'') \subseteq F'$. So we can define the composite to be the pair $(E'', g' \circ f'|_{E''})$.

Exercise 6. Show that the composite is well defined.

This gives us a category, which we will call $CW\mathsf{Sp}$. This is the category of CW spectra.

Exercise 7. Show that the inclusion of a cofinal subspectrum is an isomorphism in this category.

Exercise 8. Show that for a CW spectrum E, and a cofinal subspectrum F, there is a one-to-one correspondence between their stable cells.

Exercise 9. We have defined π_* as a functor on spectra where the morphisms are *functions*. We obviously want this definition to descend to maps. Show that if $f : E \to F$ is a *map* of CW spectra, then we get a well-defined morphism $\pi_*(f)$ on homotopy groups.

Remember, we are trying to concoct the stable homotopy category. We would like to say that $SHC = \mathsf{ho}\,\mathsf{Sp}$, but we need to have a notion of homotopy to do that.

Definition 3.21. Let K be a CW complex and let E be a CW spectrum. Define $K \wedge E$ to be a the CW spectrum $(K \wedge E)_n = K \wedge E_n$, where the structure maps are the obvious inclusions.

This actually defines a functor

$$\mathsf{CWcx} \times CW\mathsf{Sp} \to CW\mathsf{Sp}.$$

The term one sometimes uses is that $CW\mathsf{Sp}$ is tensored in CW complexes. Whenever you have such a structure, you automatically get a concept of homotopy. In particular, let I denote the unit interval.

Definition 3.22. Let $f, g : E \to F$ be maps. Then a *homotopy from f to g* is a map $h : E \wedge I_+ \to F$ so that the following diagram commutes

$$\begin{array}{ccccc} E \wedge \{0\}_+ & \longrightarrow & E \wedge I_+ & \longleftarrow & E \wedge \{1\}_+ \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle g} & \\ & & F & & \end{array}$$

This gives an equivalence relation on the set of maps from E to F. We denote the quotient set by $[E, F]$. We call this set the set of homotopy classes of maps from E to F. We are now in a position to define the stable homotopy category.

Definition 3.23. The *stable homotopy category*, denoted SHC is the category whose objects are the same as those of $CW\mathsf{Sp}$ and whose morphisms are the homotopy classes of maps. That is $\mathsf{SHC}(E, F) := [E, F]$ for CW spectra E and F.

Remark 3.24. There is an easy way to put a grading on the morphisms in both $CW\mathsf{Sp}$ and in SHC. Let $r \in \mathbb{Z}$ and define $f : E \to F$ to be a *function of degree r* if it is given by continuous maps $f_n : E_n \to F_{n-r}$ which commute with the structure maps. This clearly generalizes to the concept of *maps of degree r*. We can then define $\mathsf{Sp}(E, F)_r$ to be the set of degree r maps. Finally, we can consider the homotopy relation on $\mathsf{Sp}(E, F)_r$, which gives $[E, F]_r$ the homotopy classes of maps of degree r.

We really ought to give some examples of objects in this category. At this point, we can define an important functor. Consider the functor

$$\Sigma^\infty : CWcx \to \mathsf{SHC}.$$

This is defined as follows: let X be a CW complex. Define $\Sigma^\infty X$ to be the spectrum with $(\Sigma^\infty X)_n := \Sigma^n X$, with the obvious structure maps. This defines a functor. By abuse of notation, topologists typically use S^n to denote $\Sigma^\infty S^n$. In fact, this even defines a functor

$$\Sigma^\infty : CWcx \to CW\mathsf{Sp}.$$

Exercise 10. Show that $K \wedge \Sigma^\infty L \cong \Sigma^\infty(K \wedge L)$.

Now remember that one of things we wanted SHC to do was to capture stable invariants, and in particular, we wanted our functor Σ^∞ to somehow be the "universal stable invariant." To justify our category then, we wish to have the following

Proposition 3.25. *Let F be a CW spectrum, and K a finite complex. Then*

$$[\Sigma^\infty K, F]_r \cong \operatorname*{colim}_{n\to\infty}[\Sigma^{n+r}K, F_n].$$

Here, the colimit on the right is over the tower where the morphisms $[\Sigma^{n+r}K, F_n] \to [\Sigma^{n+r+1}K, F_{n+1}]$ is the composite

$$[\Sigma^{n+r}K, F_n] \xrightarrow{\Sigma} \left[\Sigma^{n+r+1}K, \Sigma F_n\right] \longrightarrow \left[\Sigma^{n+r+1}K, F_{n+1}\right].$$

In particular, we have for

$$[\Sigma^\infty S^0, F]_n = \pi_n F.$$

double check whether or not n or $-n$.

Proof. Clearly, for any n, there is a map

$$[\Sigma^{n+r}K, F_n] \to [\Sigma^\infty K, F]_r.$$

This map is defined by taking a map $f : \Sigma^{n+k}K \to F_n$ and then noting that this uniquely determines a map of spectra $\Sigma^\infty K \to F$ of degree r. This is because once we have f, the higher components are forced. Clearly if $f, g : \Sigma^{n+k}K \to F_n$ are homotopic as maps of spaces, then they become homotopic as maps of spectra. We need to check first that these maps determine a map from the colimit to $[\Sigma^\infty K, F]_r$.

Suppose that two maps $f : \Sigma^{n+r}K \to F_n$ and $g : \Sigma^{m+r}K \to F_m$ become identified in the colimit. This means that, while f and g may not be homotopic, they are homotopic after some number of suspensions. So there is an ℓ sufficiently large so that there is a homotopy

$$h_\ell : \Sigma^{\ell+r}K \wedge I_+ \to \Sigma^\ell F_n$$

which is a homotopy between $\Sigma^{\ell-n} f$ and $\Sigma^{\ell-m} g$. This yields a map

$$h_\ell : \Sigma^{\ell+r} K \wedge I_+ \to F_{n+\ell}.$$

Once we have h_ℓ, this uniquely determines a map of spectra

$$h : \Sigma^\infty K \wedge I_+ \to F$$

of degree r. It is clear that this is a homotopy from the *map of spectra* f to the *map of spectra* g. This implies that there is a unique morphism

$$\vartheta : \operatorname*{colim}_{n\to\infty} [\Sigma^{n+r} K, F_n] \to [\Sigma^\infty K, F].$$

To show that this is a bijection, let $f : \Sigma^\infty K \to F$ be some map of spectra. Now, f may be defined on some cofinal subspectrum E. But, since K is a finite complex[1], there is an N such that $E_n = \Sigma^n K$ for $n \geq N$. In particular, f is then uniquely determined by the component $f_{N+r} : \Sigma^{N+r} K \to F_N$. This shows that f is in the image of the $[\Sigma^{N+r} K, F_N]$, and hence of the colimit. We leave injectivity as an exercise for the reader. □

Corollary 3.26. *Let X be a CW complex, then $\pi_*(\Sigma^\infty X) \cong \pi_*^s(X)$.*

Remark 3.27. The functor

$$\Sigma^\infty : \mathsf{ho\,Top}_* \to \mathsf{SHC}$$

is not an embedding. In fact the map

$$[X, Y]_* \to [\Sigma^\infty X, \Sigma^\infty Y]_*$$

need note be injective (As we have seen), nor surjective (e.g. Kahn-Priddy map).

I want to say a few more words about this concept of *degree*. Now, there is an endofunctor

$$[1] : \mathsf{Sp} \to \mathsf{Sp}$$

which is defined by

$$(X[1])_n := X_{n+1},$$

with the obvious structure maps. This obviously induces a functor $CW\mathsf{Sp}$ and on SHC. This is the *shift functor*. We also have an endofunctor by shifting in the opposite direction. Namely, there is the functor

$$[-1] : \mathsf{Sp} \to \mathsf{Sp}$$

defined by

$$(X[-1])_n = X_{n-1}.$$

Exercise 11. Let X be a spectrum. Show that $\pi_n(X[1]) = \pi_{n-1} X$. Likewise, show that $\pi_n(X[-1]) = \pi_{n-1} X$.

[1] This is the first point we are using that K is a finite complex.

Clearly, composing these functors any number of times allows us to define functors

$$[n]: \mathsf{Sp} \to \mathsf{Sp}$$

defined by

$$(X[n])_k = X_{k-n}.$$

Remark 3.28. A function $f : X \to Y$ of degree r is nothing more than a function $f : X[r] \to Y$, or equivalently, a function $f : X \to Y[-r]$. What this shows is that we have an adjunction

$$\mathsf{Sp}(X[1], Y) \cong \mathsf{Sp}(X, Y[-1]).$$

This clearly induces a corresponding adjunction in $CW\mathsf{Sp}$.

Remark 3.29. We obviously have the following identities

$$X = X[1][-1] = X[-1][1].$$

Recall that we want some sort of "suspension" functor on SHC which is compatible with the suspension functor on Top_*. In light of the isomorphism

$$\pi_n(X[1]) \cong \pi_{n-1}X,$$

the functor $[1]$ exhibits the correct sort of behavior. We can also show that it is compatible with the suspension functor on spaces. Indeed, let $X \in \mathsf{Top}_*$, then the following shows the spaces in $\Sigma^\infty X, \Sigma^\infty X[1]$, and $\Sigma^\infty \Sigma X$.

	-2	-1	0	1	2
$\Sigma^\infty X$	$*$	$*$	X	ΣX	$\Sigma^2 X$
$\Sigma^\infty \Sigma X$	$*$	$*$	ΣX	$\Sigma^2 X$	$\Sigma^3 X$
$\Sigma^\infty X[1]$	$*$	X	ΣX	$\Sigma^2 X$	$\Sigma^3 X$

Now the spectra $\Sigma^\infty \Sigma X$ and $\Sigma^\infty X[1]$ are *not* equal, but, when X is a CW complex, then we clearly have that $\Sigma^\infty \Sigma X$ is a cofinal subspectrum of $\Sigma^\infty X[1]$, and hence are naturally isomorphic. In this sense, the functor $[1]$ is compatible with Σ. So it seems that this functor gives us our desired stability properties for spectra.

This however, is not quite right. We have shown that for finite complexes K and L, we have

$$[\Sigma^\infty K, \Sigma^\infty L] \cong \operatorname*{colim}_{k\to\infty} [\Sigma^k K, \Sigma^k L].$$

This shows that there is a natural abelian group structure on $[\Sigma^\infty K, \Sigma^\infty L]$, and furthermore, this structure is functorial on maps between finite complexes. We want this to extend naturally to $[X, Y]$ for *all* spectra X and

Y. The functor $[1]$ cannot do give us such a structure. A functor that can, however, is

$$\Sigma : \mathsf{Sp} \to \mathsf{Sp}, X \mapsto S^1 \wedge X.$$

This produces a map

$$[X,Y] \to [\Sigma X, \Sigma Y]$$

for all spectra X and Y. The target of this map has a natural group structure. The argument is the same as we gave in spaces. If we can show that the functor Σ is invertible, with inverse Σ^{-1}, then we have

$$[X,Y] \to [\Sigma X, \Sigma Y] \to [\Sigma^2 X, \Sigma^2 Y]$$

which are natural bijections, and so we get a natural abelian group structure on $[X,Y]$.

Remark 3.30. One would think that there is a natural function $\Sigma X \to X[1]$. WE would like to think that the nth component ought to be

$$\varepsilon_n : S^1 \wedge X_n \to X_{n+1}.$$

But this actually doesn't work. We need to be slightly careful about how we are defining the structure maps on $S^1 \wedge X$. It is defined by

$$\varphi_n : S^1_\sigma \wedge S^1 \wedge X_n \xrightarrow{\tau \wedge X_n} S^1 \wedge S^1_\sigma \wedge X_n \xrightarrow{S^1 \wedge \varepsilon_n} S^1 \wedge X_{n+1}.$$

Here I have written S^1_σ to distinguish between the circle carrying the suspension coordinate.

Now in order to have a function, we would need the following diagram to commute

$$\begin{array}{ccc} S^1_\sigma \wedge (S^1 \wedge X)_n & \xrightarrow{S^1_\sigma \wedge \varepsilon_n} & S^1_\sigma \wedge X_{n+1} \\ \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varepsilon_{n+1}} \\ S^1 \wedge X_{n+1} & \longrightarrow & X_{n+2} \end{array}$$

but this doesn't actually commute. This is because of the transposition morphism τ we used in defining φ_n. Now, what we can do, is slightly alter the maps $S^1 \wedge X_n \to X[1]_n$. If we define maps

$$f_n : S^1 \wedge X_n \to X[1]_n$$

by

$$f_n = (-1)^n \varepsilon_n,$$

then we still don't get a commutative diagram, but we do get a homotopy commutative diagram. This is because the map $-1 : S^1 \wedge S^1 \to S^1 \wedge S^1$ and $\tau : S^1 \wedge S^1 \to S^1 \wedge S^1$ are homotopic. Such things are called *pseudo-functions*.

While we still haven't obtained a function, the fact that the necessary squares homotopy commute does give us a well defined morphism on homotopy, so we get a natural homomorphism

$$\pi_* f : \pi_*(\Sigma X) \to \pi_*(X[1]).$$

Exercise 12. Show that this induced homomorphism is an isomorphism on homotopy groups.

It turns out that there is no natural stable equivalence $\Sigma X \to X[1]$, but the fact that there is an auxiliary spectrum X' with canonical maps $X' \to \Sigma X$ and $X' \to X[1]$, and these maps turn out to be stable equivalences. We will come back to this later.

4. Properties of $CW\mathsf{Sp}$ and SHC

4.1. **Coproducts.** I want to show that this category has all products and coproducts.

Definition 4.1. Let $\{E^\alpha\}_{\alpha\in A}$ be some collection of CW-spectra. Define

$$\bigvee_\alpha E^\alpha$$

to be the CW spectrum whose nth term is

$$\bigvee_\alpha E^\alpha_n.$$

This makes sense since

$$S^1 \wedge \left(\bigvee_\alpha E^\alpha_n\right) = \bigvee_\alpha S^1 \wedge E^\alpha_n.$$

I want to show that $CW\mathsf{Sp}$ has coproducts.

Proposition 4.2. *For any collection $\{E^\alpha\}_{\alpha\in A}$ of spectra, the inclusions $i_\alpha : E^\alpha \to \bigvee_\alpha E^\alpha$ induce, for any spectrum F, bijections*

$$\prod_\alpha i^*_\alpha : \mathsf{Sp}(\bigvee_\alpha E^\alpha, F) \to \prod_\alpha \mathsf{Sp}(E^\alpha, F)$$

$$\prod_\alpha i^*_\alpha : [\bigvee_\alpha E^\alpha, F] \to \prod_\alpha [E^\alpha, F].$$

Proof. I will only do this for two spectra. You can check the details in greater generality for yourself. We need to show that this morphism is both injective and surjective. To see that it is injective, suppose that $f, g : \bigvee E^\alpha \to F$ are two maps of spectra with the property that they coincide on each component E^α. Let f be represented by (H,φ) and g be represented by (K,ψ). Then we can write

$$H_n = H_n^1 \vee H_n^2$$

where H_n^i is a subcomplex of E^i. Note that these define cofinal subspectra H^i of E^i. Likewise we get cofinal subspectra G^i of E^i. The assumption then becomes that there is a cofinal subspectrum K^i of $H^i \cap G^i$ such that $f|K^i = g|K^i$. But then $K = K^1 \vee K^2$ is a cofinal subspectrum of E. Thus $f = g$. The surjetivity follows easily from the definitions. This proves the first statement.

To arrive at the second statement, just note that the bijection we have just found preserves the relation of homotopy. This is left as an exercise. □

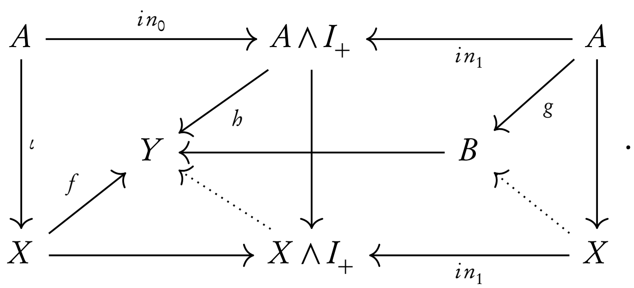
4.2. **Spectral Whitehead Theorem.** Now I want to discuss the version of the Whitehead Theorem for spectra. In particular, we will show that a stable equivalence (i.e. a π_*-isomorphism) between CW spectra, is the same as a homotopy equivalence. Towards this end, we are going to prove a spectral version of HELP.

Lemma 4.3. *Let E be a CW spectrum and G a subspectrum of E which is not cofinal. Then E has a subspectrum F such that $G \subset F \subset E$ and F contains exactly one more stable cell than G.*

Proof. As G is not cofinal, there is some stable cell e which does not belong to G. It has a representative e_n which is contained in a finite complex $K \subseteq E_n$. So there are finite subomplexes containing representatives for the stable cell e. Among such K choose the one with the fewest cells. Then $K = L \cup e$, where e is a top dimensional cell of K. Then L has fewer cells than K and so all of its cells determine stable cells of G. So there is an N such that $\Sigma^N L \subseteq G_{n+N}$. Define F to be the subspectrum obtained by adjoining the $\Sigma^r e$ for $r \geq n+N$. □

This allows us to make inductive arguments.

Lemma 4.4. *(HELP, spectral version) Let (X,A) be a pair of CW spectra, and (Y,B) a pair for which $\pi_*(Y,B)=0$. Suppose we are given a map $f: X \to Y$ and a homotopy $h: A \wedge I_+ \to Y$ from $f|_A$ to a map $g: A \to B$. Then the homotopy can be extended over $X \wedge I_+$ so as to deform f to a map into B. In* *a diagram, this is expressed as*



Proof. We work at the level of functions. Consider the collection of pairs (U,k) where $A \subseteq U \subseteq X$, i.e. U is a subspectrum of X containing A, and where $k : U \wedge I_+ \to Y$ is a function so that $k|_{U\wedge 0_+} = f|U$ and[2] $k|_{U\wedge 1_+} : U \to B$.

Observe that this set is partially ordered in the obvious way: We declare $(U,k) < (V,j)$ if $U \subseteq V$ and $j|_{U\wedge I_+} = k$. Observe further that this set is clearly inductive: If $\{(U_\lambda, k_\lambda)\}_{\lambda\in\Lambda}$ is a chain in this collection of pairs. Then if we set $U := \bigcup_\lambda U_\lambda$ and $k = \bigcup_\lambda k_\lambda$, then (U,k) is an element of our collection. So by Zorn's lemma, there is a maximal element (U',k').

We show that $U' \subseteq X$ is a cofinal spectrum. Suppose this were not the case. Then by the previous lemma, there is a subspectrum V such that $U' \subseteq V \subseteq X$ which has exactly one more cell than U'. Let n be the smallest integer where this new cell appears, i.e. $V_n = U_n \cup e^m$ and for $\ell < n$ we have $V_\ell = U_\ell$. Consider the map of pairs,

$$k'_n|_{\partial e^m \wedge I_+} : \partial e^m \wedge I_+, \partial e^m \wedge 1 \to Y_n, B_n.$$

This defines an element of $\pi_m(Y_n, B_n)$. Since $\pi_*(Y,B) = 0$, if we suspend enough times, this map becomes null. So let $N \gg 0$ so that on $V_{n+N} = U_{n+N} \cup e^{m+N}$, we have that the map

$$k'_n|_{\partial e^m \wedge I_+} : \partial e^m \wedge I_+, \partial e^m \wedge 1 \to Y_{n+N}, B_{n+N}.$$

is null homotopic. A choice of null homotopy yields an extension of k'_{n+N}

$$k''_{n+N} : V_{n+N} \wedge I_+, V_{n+N} \wedge 1_+ \to Y_{n+N}, B_{n+N}.$$

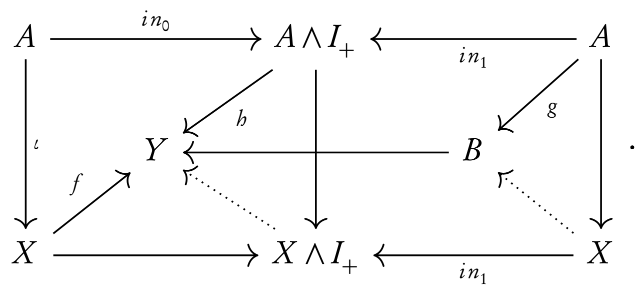
We define k''_{n+p} to be the suspension of this map for $p > N$. This produces an extension of k' to a larger subspectrum, contradicting the assumption that (U',k') was maximal. Thus U' is cofinal, and hence k' defines a map $X \wedge I_+ \to Y$. By construction, k' does what we need it to. □

[2]This may be confusing since f is only a map. In this equality, we mean equality of *maps*, i.e. that $f|U$ and $k|_{U\wedge 0_+}$ agree on a cofinal subspectrum.

Remark 4.5. A trivial alteration of this proof shows that we can allow f, g and h to be maps of degree r.

We will need the following slight generalization of HELP,

Lemma 4.6. *(generalised HELP) Let (X,A) be a pair of CW spectra, and let $\varphi : B \to Y$ be a function of spectra which induces an isomorphism on homotopy groups. Suppose we are given a map $f : X \to Y$ and a homotopy $h : A \wedge I_+ \to Y$ from $f|_A$ to a map $g : A \to B$. Then the homotopy can be extended over $X \wedge I_+$ so as to deform f to a map into B. In a diagram, this is expressed as*



Proof. The proof is similar, but we consider the triples (U,k,g') where $g' : U \to B$ such that $\varphi g' = k' in_1$. □

Exercise 13. Show that if Y is a spectrum with $\pi_* Y = 0$, then for any CW pair (X,A), any map $f : A \to Y$ can be extended to X.

We can deduce in exactly the same way as for spaces the spectral version of the Whitehead Theorem.

Theorem 4.7 (Whitehead Theorem, spectral version)**.** *Let $f : E \to F$ be a stable equivalence between spectra. Then for any CW spectrum X, the induced map*

$$f_* : [X,E]_* \to [X,F]_*$$

is a bijection.

Proof. For surjectivity, use HELP for the CW pair $(X,*)$. For injectivity use HELP for the pair $(X \wedge I_+, X \wedge (\partial I)_+)$. □

Corollary 4.8. *If $f : E \to F$ is a map of CW spectra which is a stable equivalence, then f is an isomorphism in* SHC.

We want to show that SHC is really a stable category. That is, we want to show that the suspension functor is invertible in this category. Towards that end, we need a relative version of the Whitehead theorem.

Theorem 4.9. *Let $f : (E, A) \to (F, B)$ be a function between pairs of spectra which induces an isomorphism on relative homotopy groups. Then for any CW spectrum X, the induced map*

$$f_* : [I \wedge X, X; E, A]_* \to [I \wedge X, X; F, B]_*$$

is a bijection.

Proof. Define a new spectrum R by

$$R_n := P(E_n, A_n),$$

the space of paths in E_n starting at the base point and ending somehwere in A_n. We want this to come together to form a spectrum. Define the structure maps ρ_n to be the composites

$$P(E_n, A_n) \xrightarrow{P\varepsilon'_n} P(\Omega E_{n+1}, \Omega A_{n+1}) \cong \Omega P(E_{n+1}, A_{n+1}).$$

Construct another spectrum T so that $T_n := P(F_n, B_n)$ with the structure maps defined in a similar way. Note that the funtion $f : (E, A) \to (F, B)$ induces a function $f : R \to T$. Since f induces an isomorphism on relative homotopy groups, it follows that the function $R \to T$ is a stable equivalence. The Whitehead Theorem implies that

$$[X, R] \to [X, T]$$

is an isomorphism for all CW spectra X. Unwrapping the definitions, this shows that

$$f_* : [I \wedge X, X; E, A] \to [I \wedge X, X; F, B]$$

is an isomorphism. □

4.3. **The suspension functor.** Recall that in SHC we have defined $\Sigma X := S^1 \wedge X$. This is an endofunctor of SHC and so we get a map

$$\Sigma : [X, Y]_* \to [\Sigma X, \Sigma Y]_*.$$

We wish to show that this natural map is a bijection.

Theorem 4.10. *The natural map $\Sigma : [X, Y] \to [\Sigma X, \Sigma Y]$ is a bijection.*

Proof. Observe that we have the following commutative diagram,

$$\begin{array}{ccc} [X, Y] & \longrightarrow & [I \wedge X, X; I \wedge Y, Y] \\ \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle j^Y_*} \\ [\Sigma X, \Sigma Y] & \xrightarrow{j^*_X} & [I \wedge X, X; \Sigma Y, pt] \end{array}$$

Here the maps j_X and j_Y are given by

$$j_X : I \wedge X, X \to \Sigma X, pt$$

and

$$j^Y : I \wedge Y, Y \to \Sigma Y, pt.$$

The top horizontal arrow is injective, since restriction defines a right inverse. It is also surjective.

The bottom horizontal arrow is clearly a bijection. The right hand vertical arrow is a bijection by the following colimit. ... □

So we conclude that the suspension functor defines a bijection. This means that, as we have indicated above, we have that $[X,Y]$ has a natural abelian group structure. Moreover, this abelian group structure is compatible with composition, i.e. composition is bilinear: we have homomorphisms

$$\circ : [Y,Z] \otimes_{\mathbb{Z}} [X,Y] \to [X,Z].$$

Now, I want to conclude that Σ is an invertible functor. Since Σ is stably equivalent to $[1]$ by a zig-zag, we can conclude that on $CW\mathsf{Sp}$ that there are homotopy equivalences $\Sigma X \to X[1]$, which gives us homotopy equivalences $\Sigma X[-1] \to X$. So, in a sense, the inverse of Σ is just the functor $[-1]$. The problem is that there is not a *natural* homotopy equivalence relating them. We would like to have a more natural inverse.

I should have mentioned this earlier, but I suppose later is better than never. Recall that we have a functor

$$\mathsf{Sp} \times \mathsf{Top}_* \to \mathsf{Sp}; E, X \mapsto E \wedge X,$$

where the structure maps are given by

$$\varepsilon_n \wedge S^1 : \Sigma(E \wedge X)_n = \Sigma(E_n \wedge X) = S^1 \wedge E_n \wedge X \to E_{n+1} \wedge X.$$

Note I have switched which side we are smashing on. We have another functor

$$\mathsf{Top}_* \times \mathsf{Sp} \to \mathsf{Sp}; X, E \mapsto \mathrm{Map}(X,E).$$

Here, we define

$$\mathrm{Map}(X,E)_n = \mathrm{Map}_*(X, E_n).$$

We define the structure maps of this spectrum by

$$S^1 \wedge \mathrm{Map}_*(X,E_n) \longrightarrow \mathrm{Map}_*(X, S^1 \wedge E_n) \longrightarrow \mathrm{Map}_*(X, E_{n+1}).$$

Now it turns out that for a fixed space X, we have an adjunction $- \wedge X \dashv \mathrm{Map}(X,-)$ between functors on spectra. This is because if we have a a function $f_n : E_n \wedge X \to F_n$, then the usual adjunction on spaces gives us maps $f_n' : E_n \to \mathrm{Map}_*(X, F_n)$. Remember that adjunctions have *natural* bijections between the hom-sets. This means that

$$\begin{array}{ccc} S^1 \wedge E_n \wedge X & \xrightarrow{\varepsilon_n \wedge X} & E_{n+1} \wedge X \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ S^1 \wedge F_n & \xrightarrow{\eta_n} & F_{n+1} \end{array}$$

commutes if and only if

$$\begin{array}{ccc} S^1 \wedge E_n & \xrightarrow{\varepsilon_n} & E_{n+1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ S^1 \wedge \mathrm{Map}(X, F_n) & \longrightarrow & \mathrm{Map}(X, F_{n+1}) \end{array}$$

commutes. Thus we have defined an adjunction

$$\mathsf{Sp}(E \wedge X, F) \cong \mathsf{Sp}(E, \mathrm{Map}(X, F)).$$

Keep in mind that $\mathsf{Sp}(E, F)$ is actually a topological space, this adjunction is actually a homeomorphism between spaces.

In the particular case that $X = S^1$, then we define $\Omega F := \mathrm{Map}(S^1, F)$. Thus we get a natural homeomophism

$$\mathsf{Sp}(\Sigma E, F) \cong \mathsf{Sp}(E, \Omega F).$$

Thus, as this is a homeomorphism, in the case when E And F are CW spectra, we get an adjunction

$$[\Sigma E, F] \cong [E, \Omega F].$$

Now let's return to the suspension functor. WE have shown that the natural map

$$[X, Y] \to [\Sigma X, \Sigma Y]$$

is a bijection. By the adjunction, we have that

$$[\Sigma X, \Sigma Y] \cong [X, \Omega \Sigma Y].$$

This results in the following commutative diagram

$$\begin{array}{ccc} [X, Y] & \xrightarrow{\cong} & [\Sigma X, \Sigma Y] \\ & \searrow^{(\eta_Y)_*} & \downarrow{\scriptstyle \cong} \\ & & [X, \Omega \Sigma Y] \end{array}$$

where $\eta_Y : Y \to \Omega \Sigma Y$ is the unit of the adjunction. This implies that $Y \to \Omega \Sigma Y$ is a *natural* stable equivalence (for any general spectrum Y).

When Y is a CW spectrum, then $\Omega\Sigma Y$ is also a CW spectrum, and so by Whitehead's theorem this is a natural homotopy equivalence.

Not quite, $\Omega\Sigma Y$ has its spaces CW complexes, but the structure maps needn't be embeddings. I need to fix this.

4.4. **(Co)fibre sequences.** I want to define cofibre sequences now. Let $f : E \to F$ be a map of spectra. Define the *mapping cone* of f, Cf, to be the CW spectrum given by

$$(Cf)_n := F_n \cup_{f'_n} (E'_n \wedge I)$$

where (E', f') is a representative of f. If (E'', f'') is another representative of f, then the CW spectrum $F_n \cup_{f''_n} (E''_n \wedge I)$ shares a cofinal subspectrum in common with the previous one, and so are equivalent in $CW\mathsf{Sp}$. Moreover, if we f, g are homotopic maps, then the cofibres Cf and Cg are homotopy equivalent, though this equivalence depends on the choice of homotopy relating f and g.

Definition 4.11. Let X be a CW spectrum and A a subspectrum of X. We say that A is *closed* if for any finite subcomplex $K \subseteq X_n$, if for $N \gg 0$ we have $\Sigma^N K \subseteq A_{n+N}$, then $K \subseteq A_n$.

In a simple English sentence, the subspectrum A is closed if whenever a cell eventually finds its way into A, then it was in A to begin with. Put another way, we have that A is closed in X if for any subspectrum B, if $A \subseteq B \subseteq X$ and A is cofinal in B, then $A = B$. We clearly have the following.

Proposition 4.12. *Any subspectrum of a CW spectrum can be closed.*

Exercise 14. Prove this. (Hint: Remember that cells desuspend only finitely many times.)

Exercise 15. Show that if $A \subseteq X$ is cofinal, then its closure is X.

Suppose now that $i : X \to Y$ is the inclusion of a closed subspectrum. Then we can form the spectrum Y/X. This spectrum is defined by

$$(Y/X)_n := Y_n/X_n.$$

We need to check this defines a spectrum. This follows though, since by virtue of X being closed, we have that

$$S^1 \wedge (Y_n/X_n) \cong (S^1 \wedge Y_n)/(S^1 \wedge X_n)$$

and we then get a well defined map

$$S^1 \wedge (Y_n/X_n) \to Y_{n+1}/X_{n+1},$$

and it is trivial to check this is the inclusion of a subcomplex.

Moreover, if $X \subseteq Y$ is a closed subspectrum, then we have a map

$$r : Ci = Y \cup_i CX \to Y/X$$

which is a levelwise equivalence. Thus r is a stable equivalence, and so a homotopy equivalence by the Whitehead theorem.

Now suppose that $f : X \to Y$ is a map. Then we have morphisms

$$X \xrightarrow{f} Y \xrightarrow{i} Cf = Y \cup_f CX.$$

The following proposition follows in essentially the same way as for spaces.

Proposition 4.13. *For each Z, the sequence*

$$[X,Z] \xleftarrow[f^*]{} [Y,Z] \xleftarrow[i^*]{} \left[Y \cup_f CX, Z\right]$$

Definition 4.14. A *distinguished cofibre sequence* is a sequence of morphisms of the form

$$X \xrightarrow{f} Y \xrightarrow{i} Cf = Y \cup_f CX.$$

A *cofibre sequence* is a sequence of morphisms $A \to B \to C$ for which there is a distinguished cofibre sequence $X \to Y \to Cf$ and a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ X & \longrightarrow & Y & \longrightarrow & Cf \end{array}$$

Just as in spaces, we can iteratively take cofibres to get a long cofibre sequence,

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \longrightarrow (Y \cup_f CX) \cup_i CY \longrightarrow \cdots.$$

Observe that Y is a closed subspectrum of $Y \cup_f CX$, and so we have a natural stable equivalence

$$(Y \cup_f CX) \cup_i CY \to (Y \cup_f CX) \cup_i CY/CY = \Sigma X,$$

just as in spaces. Thus we get

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \longrightarrow \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \longrightarrow \cdots,$$

where each adjuacent pairs of maps is a cofibre sequence. Now, what is amazing is that, in the stable homotopy category, cofibre sequences are the same thing as fibre sequences.

Proposition 4.15. *For any spectrum W, the sequence*

$$[W,X] \xrightarrow{f_*} [W,Y] \xrightarrow{i_*} [W,Cf]$$

Proof. Suppose we have a map $g : W \to Y$ so that $ig \simeq 0$. Let h be a null homotopy of this composite. This is precisely a map $h : CW \to Cf$. We thus have the following diagram, where the horizontal parts are cofibre sequences,

$$\begin{array}{ccccccccccc} W & \xrightarrow{id} & W & \xrightarrow{i} & CW & \longrightarrow & \Sigma W & \xrightarrow{-1} & \Sigma W & \longrightarrow & \cdots \\ \downarrow{\scriptstyle -\Sigma^{-1}k} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \Sigma g} & & \\ X & \xrightarrow{f} & Y & \xrightarrow{i} & Cf & \longrightarrow & \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y & \longrightarrow & \cdots \end{array}$$

Because these are cofibre sequences, we get the dashed arrow k making the diagram commute for free, and the rest is automatic. But Σ has an inverse in SHC. Applying Σ^{-1} gives us a map $\Sigma^{-1}k : W \to X$, making the necessary diagram commute (functoriality of Σ^{-1}). This proves exactness. □

So fibre sequences also exist in SHC are are actually just cofibre sequences. You should expect this, by the way. The stable homotopy groups π_* certainly have long exact sequences associated to fibre sequences, since unstable homotopy groups have that property. But homotopy excision tells us that the stable homotopy groups are actually a homology theory, and homology theories have long exact sequences with respect to cofibre sequences. Furthremore, homotopy excision also told us that the canonical map $\eta : \Sigma Ff \to Cf$ is a stable equivalence. So this is something we expected.

Anyway, in SHC, we have a cofibre sequence going off in both directions

$$\cdots \longrightarrow \Sigma^{-1}X \longrightarrow \Sigma^{-1}Y \longrightarrow \Sigma^{-1}Cf \longrightarrow X \longrightarrow Y \longrightarrow Cf \longrightarrow \Sigma X \longrightarrow \Sigma$$

Proposition 4.16. *Finite coproducts and finite products coincide in* SHC.

Proof. Observe that the sequence

$$X \to X \vee Y \to Y$$

is a cofibre sequence since

$$X \vee Y/X = Y.$$

This gives us an exact sequence

$$[W,X] \to [W,X \vee Y] \to [W,Y]$$

for any W. There is an obvious section to the second map coming from the inclusion $Y \to X \vee Y$. As the sets $[W,X]$, etc are actually abelian groups, we get

$$[W,X \vee Y] \cong [W,X] \oplus [W,Y] = [W,X] \times [W,Y] \cong [W,X \times Y].$$

The last isomorphism follows from the universal property of a the product. So we have, by Yoneda, a canonical isomorphism

$$X \vee Y \cong X \times Y.$$

□

4.5. **The smash product and function spectra.** This is the hardest piece of structure to grapple with in the Adams-Boardmann category, and I won't talk about it it too much. This is really a very difficult thing to get at.

What we want is some sort of functor

$$\wedge : \mathsf{Sp} \times \mathsf{Sp} \to \mathsf{Sp}$$

which makes the category of spectra Sp into a symmetric monoidal category. Here is an obvious thing you would attempt. Let X and Y be spectra which. Define a spectrum indexed on the coset $2\mathbb{Z}$ by

$$(X \wedge^{2\mathbb{Z}} Y)_{2n} := X_n \wedge Y_n.$$

There is an obvious functor

$$\mathsf{Sp}^{2\mathbb{Z}} \to \mathsf{Sp}^{\mathbb{Z}},$$

where if X is a spectrum indexed on $2\mathbb{Z}$, then we define $\widetilde{X}$ to be the spectrum given by

$$\widetilde{X}_n := X_{2n}.$$

This is clearly an equivalence of categories. So we get a functor

$$X \wedge Y := \widetilde{X \wedge^{2\mathbb{Z}} Y}.$$

Remark 4.17. This does not define a symmetric monoidal structure on Sp. In particular, we have that

- The suspension spectrum of S^0 is not a monoidal unit,
- The smash product is not associative,
- the smash product is not commutative.

By the way, this is not necessarily the only think you would attempt. You could more generally fix some functions $n, m : \mathbb{Z} \to \mathbb{Z}$ such that $n(p) + m(p) = p$ and $n(p), m(p) \to \infty$ as $p \to \infty$. We could then define a smash product by

$$P_p := X_{n(p)} \wedge Y_{m(p)}.$$

Each gives rise to a so-called "handicrafted smash product." They all fail to make the category of spectra into a symmetric monoidal category. However, these issues all vanish upon passing to the homotopy category. Rather than prove this (its rather tedious and difficult), let me just state the necessary results.

Theorem 4.18. *The naïve smash product on spectra defines a smash product*

$$\wedge : \mathsf{SHC} \times \mathsf{SHC} \to \mathsf{SHC}$$

so that for CW complexes X and Y there is a natural isomorphism

$$\Sigma^\infty X \wedge \Sigma^\infty Y \to \Sigma^\infty (X \wedge Y).$$

Moreover, the smash product $\wedge$ defines a symmetric monoidal structure on SHC *with unit $\Sigma^\infty S^0$. Moreover, any handi-crafted smash product gives rise to this functor $\wedge$ on* SHC.

Proposition 4.19. *Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of CW spectra and let Z be a CW spectrum. Then the natural morphism*

$$\bigvee_\alpha X_\alpha \wedge Z \to \left(\bigvee_\alpha X_\alpha \right) \wedge Z$$

is an isomorphism in SHC.

Proof. Work at the level of spectra and pick your favorite handi-crafted smash product. This is a level-wise equivalence, and hence a stable equivalence. Since any handi-crafted smash product yields $\wedge$ on SHC, this proves the proposition. □

Proposition 4.20. *If E is a spectrum and K is a space, then there is a canonical isomorphism*

$$K \wedge E \cong (\Sigma^\infty K) \wedge E.$$

Proposition 4.21. *If*

$$X \xrightarrow{f} Y \xrightarrow{i} Z$$

is a cofibre sequence and W is a spectrum, then

$$W \wedge X \to W \wedge Y \to W \wedge Z$$

is also a cofibre sequence.

Proof. It suffices to check the case when $f : X \to Y$ is an inclusion of a closed subspectrum, $i : Y \to Z$ the canonical projection. Choose your favorite handi-crafted smash product. This now follows from the usual fact on spaces that taking cofibres and smashing with a space commute. □

Once we get a symmetric monoidal structure, we also get a function spectrum. This is a functor

$$F : \mathsf{SHC}^{op} \times \mathsf{SHC} \to \mathsf{SHC}$$

and is denoted as $F(X,Y)$, this is the function spectrum. Is defined so that $F(X,-)$ is the right adjoint to $- \wedge X$, i.e.

$$[Y \wedge X, Z] \cong [Y, F(X,Z)]$$

for all X, Y, Z. This implies that

$$\pi_* F(X,Y) = [X,Y]_*.$$

So how do we prove the existence of such a spectrum $F(X,Y)$. Well, we could consider the functor

$$\mathsf{Sp}^{op} \to \mathbb{Z}\mathsf{Ab}; Y \mapsto [Y \wedge X, Z].$$

This turns out to define a cohomology theory because $- \wedge X$ preserves cofibre sequences. Thus, by the Brown representability theory, this is represented by a spectrum $F(X,Z)$ in the *stable homotopy category*. Constant and judicious use of the Yoneda lemma also shows that this defines a functor

$$F : \mathsf{SHC}^{op} \times \mathsf{SHC} \to \mathsf{SHC}.$$

The Brown representability theorem actually gives us natural isomorphisms

$$[Y \wedge X, Z] \cong [Y, F(X,Z)]$$

so we actually have defined F so that it defines a right adjoint to $\wedge$.

Exercise 16. Show that F preserves fibre sequences in either variable.

4.6. **Cellularization.** Now what we have done is construct SHC, but this was taken to be $\mathsf{ho}(CW\mathsf{Sp})$. But we want to to be able to construct spectra X and think of this as determining an object in SHC. So we need a functorial way of assigning to each spectrum X, a CW spectrum X' and a stable equivalence $X' \to X$ (or maybe the other way around). We take care of this through something called the telescope construction.

Definition 4.22. Let $E \in \mathsf{Sp}$ be a spectrum with structure maps $\sigma_k : \Sigma E_k \to E_{k+1}$. Define a spectrum $Tel(E)$ by defining

$$Tel(E)_n := (E_n \wedge n_+) \wedge \left(\bigvee_{k<n} \Sigma^{n-k} E_k \wedge [k, k+1]_+ \right) / \sim$$

where $\sim$ is the equivalence relation defined by

$$(e, k+1) \sim (\sigma_k e, k+1)$$

where $e \in \Sigma E_k$. The structure maps

$$\Sigma Tel(E)_n \to Tel(E)_{n+1}$$

are given by the obvious inclusion.

Remark 4.23. We have given $Tel(E)_n$ the weak topology. Also, it is clear from the construction that the structure maps are cofibrations. Indeed there is an obvious homeomorphism

$$Tel(E)_{n+1} = Tel(E)_n \cup_{\Sigma E_n} M_{\sigma_n}.$$

Proposition 4.24. *The projection maps $q_n : Tel(E)_n \to E_n$ define a function of spectra. Moreover, the inclusions $i_n : E_n \to Tel(E)_n$ don't define a function of spectra, but they are homotopy inverses to q_n. This shows that $q : Tel(E) \to E$ is a levelwise equivalence, and hence a stable equivalence. Also, this construction defines a functor.*

Corollary 4.25. *The function $q : Tel(E) \to E$ is a levelwise equivalence, and so a stable equivalence.*

Now recall that there is a functor

$$\Gamma : \mathsf{Top}_* \to \mathsf{Top}_*$$

and a natural weak equivalence

$$\Gamma \implies 1$$

so that ΓX is a CW complex for each space X. This clearly extends to a functor

$$\Gamma : \mathsf{Sp} \to \mathsf{Sp}$$

and a natural levelwise equivalence

$$\Gamma \implies 1.$$

This has the property that if E is a spectrum, then ΓE is also a spectrum, but with each $(\Gamma E)_n$ a CW spectrum. Thus, $Tel(\Gamma E)$ is a CW spectrum and we have natural levelwise equivalences

$$Tel(\Gamma E) \to \Gamma E \to E$$

and so the composite is a levelwise equivalence. Therefore, up to natural levelwise equivalence, each spectrum can be replaced by a CW spectrum, and hence provides an object (in a natural way) in SHC.

4.7. **Telescopes (or the homotopy colimit).** I want to just mention a way of defining colimits in SHC. For the moment lets work in spaces. Suppose that we have a tower

$$X_0 \to X_1 \to X_2 \to \cdots$$

of pointed CW complexes. Define the *telescope* (or the *homotopy colimit*) to be the the space

$$\text{hocolim}(X_\bullet) := \bigvee_{n\in\mathbb{N}} X_n \wedge [n, n+1]_+ / \sim$$

where $\sim$ is the equivalence relation generated by identifying $X_n \wedge \{n+1\}_+$ with $X_{n+1} \wedge \{n+1\}_+$ via the map $X_n \to X_{n+1}$.

Lemma 4.26. *If the tower $X_\bullet$ is a tower of inclusions of subcomplexes, then there is a canonical map*

$$\text{hocolim}\, X_\bullet \to \text{colim}\, X_\bullet$$

and this map is a homotopy equivalence.

Proof. Observe that

$$\widetilde{X}_k \bigvee_{n=0}^{k} X_n \wedge [n, n+1]_+ / \sim$$

is homotopy equivalent to X_k. Now $\text{colim}\, X_\bullet = \bigcup_k X_k$. Similarly, we have that

$$\widetilde{X}_0 \to \widetilde{X}_1 \to \cdots$$

is a tower of cofibrations, and so

$$\text{colim}_k \widetilde{X}_k = \bigcup_k \widetilde{X}_k = \text{hocolim}\, X_\bullet.$$

In particular, the projections $\widetilde{X}_k \to X_k$ determine a map in the colimit, giving

$$\text{hocolim}\, X_\bullet \to \text{colim}\, X_\bullet.$$

Observe that

$$\pi_* \,\text{hocolim}\, X_\bullet = \text{colim}\, \pi_* \widetilde{X_k} = \text{colim}\, \pi_* X_k$$

thus the map is a weak equivalence. Since $\text{hocolim}\, X_\bullet$ and $\text{colim}\, X_\bullet$ are both CW complexes, it follows from Whitehead's theorem that they are homotopy equivalent. □

I want to end this section discussing the so-called *tautological presentation* of a spectrum E. Suppose that E is a CW spectrum. Here, I will write $\Sigma^n X$ for the shift functors.

4.8. **Ω-spectra and the functor Ω^∞.** First, lets figure out what stable equivalences between Ω-spectra are.

Definition 4.27. A spectrum E is an Ω-spectrum if the adjoint of each of the structure maps $\varepsilon'_n : E_n \to \Omega E_n$ is a weak equivalence.

Lemma 4.28. *If E is an Ω-spectrum, then*

$$\pi_s E = \pi_{s+k} E_k$$

for all k so that $s + k \geq 0$.

Proof. This is because the homotopy group is defined by the following colimit

$$\pi_s E = \operatorname*{colim}_{k \to \infty} \pi_{k+s} E_k.$$

As E is an Ω-spectrum, all of the morphisms in this colimit are isomorphisms. □

Proposition 4.29. *A function $f : E \to F$ between two Ω spectra is a stable equivalence if and only if f is a levelwise equivalence.*

Proof. This follows because the isomorphisms of the previous lemma are natural in E. □

Proposition 4.30. *Suppose that $g : E \to F$ is a levelwise equivalence between spectra. Then E is an Ω-spectrum if and only if F is.*

Proof. We have the commutative diagrams

$$\begin{array}{ccc} E_n & \xrightarrow{g_n} & F_n \\ \downarrow{\scriptstyle \varepsilon'_n} & & \downarrow{\scriptstyle \varepsilon'_n} \\ \Omega E_{n+1} & \xrightarrow{\Omega g_n} & \Omega F_{n+1} \end{array}$$

As g is a levelwise equivalence, both of the horizontal arrows are weak equivalences. Thus one of the vertical arrows is a weak equivalence if and only if the other is. □

Theorem 4.31. *For a spectrum E there is an Ω-spectrum E' and a stable equivalence $E \to E'$. Furthermore, this construction is functorial and produces a natural stable equivalence $1 \implies ()'$.*

Proof. For an integer n, we have the following tower

$$E_n \to \Omega E_{n+1} \to \cdots \to \Omega^k E_{n+k} \to \cdots$$

where the maps are the adjoints of the structure maps of E. We define

$$E'_n := \operatorname*{hocolim}_k \Omega^k E_{n+k}.$$

Because S^1 is compact, we have an equivalence

$$\Omega E'_n = \operatorname{hocolim}_k \Omega^{k+1} E_{n+k}$$

so we get an induced morphism $E'_n \to \Omega E'_{n+1}$. Thus the spaces E'_n form a spectrum E'. A cofinal subdiagram argument shows that the maps $E'_n \to \Omega E'_n$ are weak equivalences, and so E' is an Ω-spectrum.

We obviously have a map $E_n \to E'_n$ for each n, I leave it as an exercise to check that these maps form a function and that this function is a stable equivalence. □

We need to have this Ω-ification construction in order to define the functor

$$\Omega^\infty : \mathsf{SHC} \to \mathsf{ho\,Top}_*.$$

The idea behind this functor is that it should send a spectrum E to its *underlying infinite loop space*. In particular, we want it to have the property that if $f : E \to F$ is a stable equivalence then $\Omega^\infty f$ is a weak equivalence. The most obvious candidate for Ω^∞ is to define

$$\Omega^\infty E := E_0,$$

but this doesn't work in general since a function $f : E \to F$ can be a stable equivalence without inducing a weak equivalence on the 0th spaces. However, if we replace E by E', then we get a sensible functor.

Remark 4.32. What I am implicitly thinking about here is that we need to consider the right derived functor (in the sense of Quillen) of $E \mapsto E_0$.

Remark 4.33. In order to really define Ω^∞ we really need to make sure that if we have a CW spectrum E then E' remains a CW spectrum. There is a modification of this construction which does that. (cf. Rudyak).

4.9. **Homology and cohomology.** We can naturally extend homology and cohomology to spectra.

Theorem 4.34 (Adams)**.** *The obvious analog of Brown representability for functors on the category of CW spectra with morphisms of degree 0 holds.*

Given a spectrum E, we can also define an associated homology and cohomology theory. Let X be a spectrum, then we define

Definition 4.35. Let E and X be spectra. Then we define the *E-homology of X* to be

$$E_n(X) := \pi_n(E \wedge X)$$

and we define the *E-cohomology of X* to be

$$E^n(X) := [X, \Sigma^n E] = [X, E]_{-n} = \pi_{-n} F(X, E).$$

This clearly gives us a means to define the reduced homology or cohomology theory associated to E on spaces. Namely, given a CW complex K, we define

$$\widetilde{E}_*(K) := \pi_*(E \wedge \Sigma^\infty K),$$

and

$$\widetilde{E}^*(X) := [\Sigma^\infty X, E]_{-*}.$$

Exercise 17. Show that these functors satisfy the Eilenberg-Steenrod axioms, and hence really do define (co)homology theories on spaces.

What is really beneficial about this perspective is that we can now think of expression E_*X is a functor in two variables, E and X.

Proposition 4.36. *Given cofibre sequences*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

and

$$E \xrightarrow{\varphi} F \xrightarrow{\psi} G$$

then we get long exact sequences

$$E_n X \xrightarrow{f_*} E_n Y \xrightarrow{g_*} E_n Z$$

and

$$E_n X \xrightarrow{\varphi_*} F_n X \xrightarrow{\psi_*} G_n X.$$

There are dual exact sequences for cohomology.

Proof. Left as an exercise. This essentially comes from using several properties about cofibre sequences. □

Proposition 4.37. *For any E and any X in* SHC*, we have natural isomorphisms*

$$E_n X \cong E_{n+1} \Sigma X$$

$$E^n(X) \cong E^{n+1}(\Sigma X)$$

and

$$E_n(S) = E^{-n} S = \pi_n E.$$

Exercise 18. Suppose that E is an Ω-spectrum. Show that if $X = \Sigma^\infty K$ for a complex K, then

$$E^n(X) \cong [K, E_n].$$

5. Examples of spectra

5.1. **Eilenberg-MacLane spectra.** The first example of a (co)homology that one sees is singular cohomology. Let G be an abelian group. Recall that we have the *Eilenberg-MacLane spaces* $K(G,n)$ with the property that they are CW complexes and

$$\pi_* K(G,n) = \begin{cases} G & * = n \\ 0 & * \neq 0 \end{cases}$$

Furthermore, these spaces have the property that they are unique up to homotopy equivalence. Also, note that

$$\pi_* \Omega K(G,n) = \pi_{*+1} K(G,n),$$

which shows that $\Omega K(G,n) \simeq K(G,n-1)$. Thus the spaces $\{K(G,n)\}_{n\in\mathbb{N}}$ come together to form an Ω-spectrum. We denote the resulting spectrum by HG.

We have already seen that if X is a space then

$$\widetilde{H}^n(X;G) \cong [X, K(G,n)]$$

and hence we have

$$\widetilde{H}^n(X;G) = [\Sigma^\infty K, HG]_{-n}.$$

I didn't mention it before, but it is true that

$$\widetilde{H}_n(X;G) \cong \operatorname*{colim}_k \pi_{n+k}(X \wedge K(\pi,k)).$$

The way you prove such a natural isomorphism is by showing the right hand side defines a reduced homology theory on spaces which satisfies the dimension axiom (take a look at May's book Chapter 22 for a proof).

In particular, unraveling the definitions then tells us that

$$\widetilde{H}_n(X;G) \cong \pi_n(\Sigma^\infty X \wedge HG) = \pi_n(X \wedge HG)$$

Now since the Eilenberg-MacLane spaces can be constructed functorially, we essentially have a functor

$$H : \mathsf{Ab} \to \mathsf{SHC}; G \mapsto HG.$$

In particular, to the map $\cdot 2 : \mathbb{Z} \to \mathbb{Z}$ we have a corresponding map

$$H\mathbb{Z} \xrightarrow{2} H\mathbb{Z}$$

of Eilenberg-MacLane spectra. Now we could consider the cofibre C of this map. This induces a long exact sequence in homotopy groups

$$\pi_* H\mathbb{Z} \xrightarrow{2} \pi_* H\mathbb{Z} \longrightarrow \pi_* C.$$

This shows that the only nontrivial homotopy group of C is π_0, which is a $\mathbb{Z}/2$. Thus $C \simeq H\mathbb{F}_2$. In other words, we have a cofibre sequence

$$H\mathbb{Z} \xrightarrow{2} H\mathbb{Z} \longrightarrow H\mathbb{Z}/2$$

in spectra. This produces a long exact sequence

$$\cdots \longrightarrow H\mathbb{Z}_*X \xrightarrow{\cdot 2} H\mathbb{Z}_*X \longrightarrow H\mathbb{Z}/2_*X \xrightarrow{\partial} H\mathbb{Z}_{*-1}X \longrightarrow \cdots.$$

In the case when $X = \Sigma^\infty K$ for a space K, this long exact sequence reduces to the usual one.

Exercise 19. Generalize the above argument to show that the functor H takes short exact sequences of abelian groups to cofibre sequences of spectra.

Similarly, we have a cofibre sequence

$$H\mathbb{Z}/2 \longrightarrow H\mathbb{Z}/4 \longrightarrow H\mathbb{Z}/2$$

arising from the short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

Again, this induces a long exact sequence

$$\cdots \longrightarrow H\mathbb{Z}/2_*X \longrightarrow H\mathbb{Z}/4_*X \longrightarrow H\mathbb{Z}/2_*X \xrightarrow{\partial} H\mathbb{Z}/2_{*-1}X \longrightarrow \cdots$$

and again when $X = \Sigma^\infty K$ for some space K, the connecting homomorphism ∂ is often referred to as the *Bockstein homomorphism*. Now the connecting homomorphism is actually given by a spectrum level map, namely,

$$\beta : H\mathbb{Z}/2 \to \Sigma H\mathbb{Z}/2,$$

obtained from continuing the cofibre sequence. Note that

$$\pi_n(\Sigma H\mathbb{Z}/2 \wedge X) = \pi_{n-1}(H\mathbb{Z}/2 \wedge X),$$

so the degrees match up. In general, the Bockstein homomorphism is nontrivial. Thus, we have found a nontrivial element in

$$(H\mathbb{Z}/2)^1 H\mathbb{Z}/2 = [H\mathbb{Z}/2, \Sigma H\mathbb{Z}/2].$$

This is the first example of an element in something called the *Steenrod algebra*. We will talk about this extensively a little later in the course.

5.2. **topological K-theory spectra.** Lets start with complex K-theory. First, I will define a functor

$$K^0 : \mathsf{Top}_* \to \mathsf{Ab}.$$

Consider the set of Vect(X) of all finite rank complex vector bundles on X up to isomorphism. For a vector bundle E on X, we let $[E]$ denote its isomorphism class. This is clearly a semi-ring, with addition and multiplication induced by the operations $\oplus$ and $\otimes$ respectively. We define $K^0(X)$ to be the Grothendieck completion of Vect(X).

More explicitly, we define $K(X)$ to be the free abelian group $F(X)$ generated by the set Vect(X), and then mod out by the relations $[E]+[F]-[E\oplus F]$. This makes $K(X)$ an abelian group. One also gets a multiplication by defining

$$[E][F] := [E \otimes F]$$

and then extending in the usual way. Another way of constructing this group is to consider the diagonal map

$$\Delta : \mathrm{Vect}(X) \to \mathrm{Vect}(X) \times \mathrm{Vect}(X)$$

and then define $K(X)$ to be the $\Delta(\mathrm{Vect}(X))$ cosets of $\mathrm{Vect}(X) \times \mathrm{Vect}(X)$. The idea is that the expressions $([E],[F])$ are formally thought of as $[E]-[F]$. These give the same group.

Recall that given a vector bundle F, there exists a vector bundle G such that $F\oplus E = \varepsilon^n$, where ε^n denotes the trivial bundle of rank n. Thus, given an expression $[E]-[F]$ we can always find a bundle G so that $F \oplus G = \varepsilon^n$ and

$$[E]-[F] = [E]-[F]+[G]-[G] = [E\oplus G]-[\varepsilon^n] = [E\oplus G]-\underline{n}.$$

Thus every element of $K^0(X)$ can be expressed as $[E]-n$ for some vector bundle E and some natural number n.

Now suppose that E and F are vector bundles on X and suppose that $[E]=[F]$ in $K^0(X)$. Then by the definition, there is a bundle G such that

$$E \oplus G \cong F \oplus G.$$

Thus, there is a bundle G' such that $G \oplus G' \cong \underline{n}$ for some n.

$$E \oplus \underline{n} \cong F \oplus \underline{n}$$

for somenatural number n.

We often call elements of $K^0(X)$ *virtual vector bundles*. There is an obvious homomorphism

$$\deg : K^0X \to \mathbb{Z}$$

defined by taking

$$[E]-[F] \mapsto \dim E - \dim F.$$

This integer is called the *virtual dimension*.

Theorem 5.1.

5.3. **Thom spectra.** Now lets talk about Thom spectra. This is an example of spectra which don't arise as an Ω-spectrum. They naturally arise as sequential spectra.

5.4. **Moore spectra.** I want to now talk about an important class of spectra known as Moore spectra. Suppose that $x \in \pi_k S$ is a nontrivial element in the stable homotopy groups of the sphere. This corresponds to a map

$$S^k \xrightarrow{x} S^0,$$

the cofibre is referred to as the cone on x, and is denoted as S/x. In the particular case when $x \in \pi_0 S = \mathbb{Z}$, then we have that $x = n$ for some integer n. In this case we refer to S/n as the *mod n Moore spectrum*.

Remark 5.2. The mod n Moore spectra are very strange. In particular, we *DO NOT* have that $\pi_*(S/n) = \pi_*(S)/n$.

Let G be an abelian group. Fix a free resolution of G,

$$0 \to R \to F \to G \to 0.$$

Let $\bigvee_{\alpha\in A} S$ and $\bigvee_{\beta\in B} S$ be such that

$$\pi_0(\bigvee_{\alpha\in A} S) = R$$

and

$$\pi_0(\bigvee_{\beta\in B} S) = F.$$

Let $i : \bigvee_{\alpha\in A} S \to \bigvee_{\beta\in B} S$ be a map of spectra inducing the map $R \to F$. Let MG denote the cofibre of i. Then we have the following.

Proposition 5.3. *We have for MG*

1. $\pi_r MG = 0$ *for* $r < 0$, *i.e.* MG *is connective,*
2. $\pi_0 MG = H_0 MG = G$, *and*
3. $H_r(MG) = 0$ *for* $r > 0$.

So we are justified in calling MG a Moore spectrum of type G. Notice that, since smashing with $H\mathbb{Z}$ preserves cofibre sequences, we have a cofibre sequence

$$H\mathbb{Z} \wedge \bigvee_{\alpha\in A} S \longrightarrow H\mathbb{Z} \wedge \bigvee_{\beta\in B} S \longrightarrow H\mathbb{Z} \wedge MG.$$

Considering the long exact sequence in π_* shows that $H\mathbb{Z} \wedge MG \simeq HG$. In particular, for example, we have $H\mathbb{Z} \wedge S/2 \simeq H\mathbb{Z}/2$.

6. Duality and orientations?? (time permitting)

7. Cohomology Operations and Homology co-operations

7.1. Ring spectra. Since we have a smash product we can define ring spectra. The idea is to take the usual diagrams defining rings in the usual sense, and replace everything with spectra and the smash product.

7.2. Operations and co-operations.

7.3. Steenrod algebra and its dual.

7.4. Adams operations.

8. The Adams spectral sequence

So how do we actually compute things? Well, one very general way to approach computations is through something called the Adams spectral sequence.

8.1. Constructing the spectral sequence.

8.2. Convergence: a digression on Bousfield localizations.

8.3. The E_2-term?

8.4. The May spectral sequence.

8.5. Hopf invariant one elements and the Kervaire elements.

9. Applications

9.1. The calculation of $\pi_* MO$ and $\pi_* MU$.

9.2. Hopf invariant one.

9.3. Vector fields on spheres.

9.4. Image of the stable J-homomorphism.

10. Vistas

10.1. Homotopie Chromatique.

10.2. autre modeles du spectres.

10.3. la conjecture de Adams.

Appendix A. Spectral sequences

Appendix B. Simplicial sets and classifying spaces

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